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HAMILTON-JACOBI TREATMENT OF CONSTRAINT FIELD SYSTEMS

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Abstract

Motivated by the Hamilton–Jacobi approach of fields with constraints, we analyse the classical structure of three different constrained fields systems: (i) the scalar field coupled to two flavours of fermions through Yukawa couplings (ii) the scalar field coupled minimally to the vector potential (iii) the electromagnetic field coupled to a spinor. The equations of motion are obtained as total differential equations in many variables. The integrability conditions are investigated. The second and third constrained systems are quantized using canonical path integral formulation based on the Hamilton-Jacobi treatment.

keywords: Hamiltonian-Jacobi formalism, constrained systems, path integral.

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1 Introduction

The most common method for investigating the Hamiltonian treatment of constrained systems was initiated by Dirac [1,2]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Besides, he showed that the number of degrees of freedom of the dynamical system can be reduced. Hence, the equations of motion of a constrained system are obtained in terms of arbitrary parameters. Moreover, the Dirac approach is widely used for quantizing the constrained Hamilton systems. The path integral is another approach used for the quantization of constrained systems of classical singular theories, which was initiated by Faddeeve [3]. Faddeeve has applied this approach when only first-class constraints in the canonical gauge are present. Senjanovic [4] generalized Faddeev's method to second-class constraints. Fradkin and Vilkovisky [5,6] red-rived both results in a broader context, where they improved the procedure to the Grassman variables. Gitman and Tyutin [7] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge.

The canonical method (or Güler's method) developed Hamilton-Jacobi formulation to investigate constrained systems [8-9]. The starting point of the Hamilton-Jacobi approach [10-14] is the variational principle. The Hamiltonian treatment of constrained systems leads us to the equations of motion as total differential equations in many variables. The equations are integrable if the corresponding system of partial differential equations is a Jacobi system. In Ref. [15] Güler has presented a treatment of classical fields as constrained systems. Then Hamilton-Jacobi quantization of finite dimensional system with constraints was investigated in Ref. [16]. The advantages of the Hamilton-Jacobi formalism [17-21] are that there are no differences between first- and second-class constraints and no need for a gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations. The Hamilton-Jacobi approach treats the constrained system as the same as Dirac's methods but in a simple way. Both methods give the same results as seen in Refs. [22-26].

This work is organized as follows: In Sec.2 Hamilton-Jacobi formula-

tion is presented. In Sec.3 the Hamilton-Jacobi formulation of the scalar field coupled to two flavours of fermions through Yukawa couplings is investigated. In Sec.4 the path integral quantization of the scalar field coupled minimally to the vector potential is obtained In Sec.5 the system as the electromagnetic field coupled to a spinor is quantized using Hamilton-Jacobi formulation. In Sec.4 The conclusions are given.

2 Hamilton-Jacobi Formulation

One starts from singular Lagrangian $L \equiv L(q_i, \dot{q}_i, t)$, $i = 1, 2, \ldots, n$, with the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j}, \tag{1}$$

of rank $(n-r)$, $r < n$. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n-r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \qquad \mu = n-r+1, \ldots, n. \tag{3}$$

where q_i are divided into two sets, q_a and x_μ. Since the rank of the Hessian matrix is $(n-r)$, one may solve Eq. (4) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{x}_\mu, p_a; t). \tag{4}$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t). \tag{5}$$

The canonical Hamiltonian H_0 reads

$$H_0 = -L(q_i, \dot{x}_\nu, \dot{q}_a; t) + p_a \dot{q}_a - \dot{x}_\mu H_\mu, \qquad \nu = 1, 2, \ldots, r. \tag{6}$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha\left(x_\beta,\ q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial x_\beta}\right) = 0, \qquad \alpha, \beta = 0, 1, \ldots, r, \tag{7}$$

where

$$H'_0 = p_0 + H_0 \ , \tag{8}$$

$$H'_\mu = p_\mu + H_\mu \ . \tag{9}$$

We define $p_\beta = \partial S[q_a; x_a]/\partial x_\beta$ and $p_a = \partial S[q_a; x_a]/\partial q_a$ with $x_0 = t$ and S being the action.

Now the total differential equations are given as

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dx_\alpha, \tag{10}$$

$$dp_a = \frac{\partial H'_\alpha}{\partial q_a} dx_\alpha, \tag{11}$$

$$dp_\beta = \frac{\partial H'_\alpha}{\partial t_\beta} dx_\alpha, \tag{12}$$

$$dz = \left(- H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dx_\alpha, \tag{13}$$

where $Z = S(x_\alpha, q_a)$. These equations are integrable if and only if [27]

$$dH'_0 = 0, \tag{14}$$

$$dH'_\mu = 0, \qquad \mu = 1, 2, \ldots, r. \tag{15}$$

If conditions (14), (15) are not satisfied identically, one considers them as a new constants and a gain consider their variations. Thus, repeating this procedure, one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the set of canonical phase space coordinates (q_a, p_a) as functions of t_a; the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters t_α, respectively. In this case the path integral representation can be written as [28, 29].

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a \exp \left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \tag{16}$$

$$a = 1, , \ldots, n-p, \qquad \alpha = 0, n-p+1, \ldots, n.$$

In fact, this path integral is an integration over the canonical phase space coordinates (q^a, p^a).

3 Hamilton-Jacobi formulation of the scalar field coupled to two flavours of fermions through Yukawa couplings

We consider one loop order the self-energy for the scalar field φ with a mass m, coupled to two flavours of fermions with masses m_1 and m_2, coupled through Yukawa couplings described by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}\lambda\varphi^3 + \sum_i \overline{\psi}_{(i)}(i\gamma^\mu\partial_\mu - m_i)\psi_{(i)}$$
$$- g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}), \qquad \mu = 0,1,2,3, \quad (17)$$

where λ is parameter and g constant, φ, $\psi_{(i)}$, and $\overline{\psi}_{(i)}$ are odd ones. We are adopting the Minkowski metric $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$.

The Lagrangian function (17) is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial\dot{q}_i\,\partial\dot{q}_j}, \quad (18)$$

is one.

The generalized momenta (2,3)

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \partial^0\varphi, \quad (19)$$

$$p_{(i)} = \frac{\partial L}{\partial\dot{\psi}_{(i)}} = i\overline{\psi}_{(i)}\gamma^0 = -H_{(i)}, \qquad i = 1,2, \quad (20)$$

$$\overline{p}_{(i)} = \frac{\partial L}{\partial\dot{\overline{\psi}}_{(i)}} = 0 = -\overline{H}_{(i)}. \quad (21)$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual $\psi_{(i)}$ as a column vector and $\overline{\psi}_{(i)}$ as a row vector implies that $p_{(i)}$ will be a row vector while $\overline{p}_{(i)}$ will be a column vector.

Since the rank of the Hess matrix is one, one may solve (19) for $\partial^0\varphi$ as:

$$\partial^0\varphi = p_\varphi \equiv \omega. \quad (22)$$

The usual Hamiltonian H_0 is given as:

$$H_0 = -L + \omega p_\varphi + \partial_0\psi_{(i)}\, p_{(i)}\Big|_{p_{(i)}=-H_{(i)}} + \partial_0\overline{\psi}_{(i)}\, \overline{p}_{(i)}\Big|_{\overline{p}_{(i)}=-\overline{H}_{(i)}}, \tag{23}$$

or

$$H_0 = \frac{1}{2}({p^2}_\varphi - \partial_a\varphi\partial^a\varphi) + \frac{1}{2}m^2\varphi^2 + \frac{1}{6}\lambda\varphi^3 - \overline{\psi}_{(i)}(i\gamma^a\partial_a - m_i)\psi_{(i)} + g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}), \qquad a = 1, 2, 3. \tag{24}$$

Eqs. (20), and (21) lead to the primary constraints.

By using Hamilton-Jacobi, the set of (HJPDE) (8) read as

$$H_0' = p_0 + H_0 = p_0 + \frac{1}{2}({p^2}_\varphi - \partial_a\varphi\partial^a\varphi) + \frac{1}{2}m^2\varphi^2 + \frac{1}{6}\lambda\varphi^3 - \overline{\psi}_{(i)}(i\gamma^a\partial_a - m_i)\psi_{(i)} + g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}), \tag{25}$$

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\,\overline{\psi}_{(i)}\,\gamma^0 = 0, \tag{26}$$

$$\overline{H}'_{(i)} = \overline{p}_{(i)} + \overline{H}_{(i)} = \overline{p}_{(i)} = 0. \tag{27}$$

Therefore, the total differential equations for the characteristic (10), (11) and (12) are:

$$d\varphi = p_\varphi d\tau, \tag{28}$$

$$d\psi_{(i)} = d\psi_{(i)}, \tag{29}$$

$$d\overline{\psi}_{(i)} = d\overline{\psi}_{(i)}, \tag{30}$$

$$dp_\varphi = \left[m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)})\right]d\tau, \tag{31}$$

$$dp_{(1)} = \left[\overline{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\,\varphi\,\overline{\psi}_{(2)}\right]d\tau, \tag{32}$$

$$dp_{(2)} = \left[\overline{\psi}_{(2)}(i\overleftarrow{\partial_a}\gamma^a + m_2) + g\,\varphi\,\overline{\psi}_{(1)}\right]d\tau, \tag{33}$$

$$d\overline{p}_{(1)} = \left[-(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)}\right]d\tau - i\gamma^0 d\psi_{(1)}, \tag{34}$$

$$d\overline{p}_{(2)} = \left[-(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)}\right]d\tau - i\gamma^0 d\psi_{(2)}. \tag{35}$$

The integrability conditions ($dH'_\alpha = 0$) imply that the variation of the constraints $H'_{(i)}$ and $\overline{H}'_{(i)}$ should be identically zero, that is

$$dH'_{(i)} = dp_{(i)} - i\,d\overline{\psi}_{(i)}\,\gamma^0 = 0, \tag{36}$$

$$d\overline{H}'_{(i)} = d\overline{p}_{(i)} = 0. \tag{37}$$

The following equations of motion:
From Eq. (28), we obtain

$$\dot{\varphi} = p_\varphi. \tag{38}$$

Substituting from Eqs. (32) and (33) into Eq. (36), we get

$$i\partial_0\overline{\psi}_{(1)}\gamma^0 - \overline{\psi}_{(1)}(i\overleftarrow{\partial_a}\gamma^a + m_1) - g\varphi\,\overline{\psi}_{(2)} = 0, \tag{39}$$

$$i\partial_0\overline{\psi}_{(2)}\gamma^0 - \overline{\psi}_{(2)}(i\overleftarrow{\partial_a}\gamma^a + m_2) - g\varphi\,\overline{\psi}_{(1)} = 0. \tag{40}$$

Substituting from Eqs. (34) and (35) into Eq. (37), we have

$$(i\gamma^\mu\partial_\mu - m_1)\psi_{(1)} - g\,\varphi\,\psi_{(2)} = 0, \tag{41}$$

$$(i\gamma^\mu\partial_\mu - m_2)\psi_{(2)} - g\,\varphi\,\psi_{(1)} = 0. \tag{42}$$

One notes that the integrability conditions are not identically zero, they are added to the set of equations of motion.
From Eqs.(31-33), we get the following equations of motion:

$$\dot{p}_\varphi = m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}), \tag{43}$$

$$\dot{p}_{(1)} = \overline{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\,\varphi\,\overline{\psi}_{(2)}, \tag{44}$$

$$\dot{p}_{(2)} = \overline{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\,\varphi\,\overline{\psi}_{(1)}. \tag{45}$$

Substituting from Eqs. (41) and (42) into Eqs. (34) and (35), we get

$$\dot{\overline{p}}_{(i)} = 0, \qquad i = 1, 2. \tag{46}$$

Differentiate Eq. (38) with respect to time, and making use of Eq. (43), we get

$$\ddot{\varphi} - m^2\varphi - \frac{1}{2}\lambda\varphi^2 - g(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}) = 0. \tag{47}$$

4 Path integral quantization of the scalar field coupled minimally to the vector potential

Consider the action integral for the scalar field coupled minimally to the vector potential as

$$S = \int d_4x\, L, \tag{48}$$

where the Lagrangian L is given by

$$L = -\frac{1}{4}\,F_{\mu\nu}(x)F^{\mu\nu}(x) + (D_\mu\varphi)^*(x)D^\mu\varphi(x) - m^2\varphi^*(x)\varphi(x), \tag{49}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{50}$$

and

$$D_\mu\varphi(x) = \partial_\mu\varphi(x) - ieA_\mu(x)\varphi(x). \tag{51}$$

The canonical momenta are defined as

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \tag{52}$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \tag{53}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (D_0\varphi)^* = \dot{\varphi}^* + ieA_0\varphi^*, \tag{54}$$

$$p_{\varphi^*} = \frac{\partial L}{\partial \dot{\varphi}^*} = (D_0\varphi) = \dot{\varphi} - i\,e\,A_0\,\varphi, \tag{55}$$

From Eqs. (52), (54) and (55), the velocities $\dot{A}_i, \dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i, p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \tag{56}$$

$$\dot{\varphi}^* = p_\varphi - ieA_0\varphi^*, \tag{57}$$

$$\dot{\varphi} = p_{\varphi^*} + ieA_0\varphi. \tag{58}$$

The canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 = \frac{1}{4}\,F^{ij}F_{ij} - \frac{1}{2}\,\pi_i\pi^i + \pi^i\,\partial_i A_0 + p_{\varphi^*}p_\varphi + ieA_0\varphi p_\varphi \\ - ieA_0\varphi^* p_{\varphi^*} - (D_i\varphi)^*(D^i\varphi) + m^2\varphi^*\varphi. \end{aligned} \tag{59}$$

Making use of Eqs. (7) and (9), we find for the set of HJPDE

$$H_0' = \pi_4 + H_0, \tag{60}$$

$$H' = \pi_0 + H = \pi_0 = 0, \tag{61}$$

Therefore, the total differential equations for the characteristic (10-12) obtained as

$$\begin{aligned} dA^i &= \frac{\partial H_0'}{\partial \pi_i}\,dt + \frac{\partial H'}{\partial \pi_i}\,dA^0, \\ &= -(\pi^i + \partial_i A_0)\,dt, \end{aligned} \tag{62}$$

$$dA^0 = \frac{\partial H_0'}{\partial \pi_0}\,dt + \frac{\partial H'}{\partial \pi_0}\,dA^0 = dA^0, \tag{63}$$

$$d\varphi = \frac{\partial H_0'}{\partial p_\varphi}\,dt + \frac{\partial H'}{\partial p_\varphi}\,dA^0,$$
$$= (p_{\varphi^*} + ieA_0\varphi)\,dt, \tag{64}$$

$$d\varphi^* = \frac{\partial H_0'}{\partial p_{\varphi^*}}\,dt + \frac{\partial H'}{\partial p_{\varphi^*}}\,dA^0,$$
$$= (p_\varphi - ieA_0\varphi^*)\,dt, \tag{65}$$

$$d\pi^i = -\frac{\partial H_0'}{\partial A_i}\,dt - \frac{\partial H'}{\partial A_i}\,dA^0,$$
$$= [\partial_l F^{li} + ie(\varphi^*\partial^i\varphi + \varphi\,\partial_i\varphi^*) + 2e^2A^i\varphi\varphi^*]\,dt, \tag{66}$$

$$d\pi^0 = -\frac{\partial H_0'}{\partial A_0}\,dt - \frac{\partial H'}{\partial A_0}\,dA^0,$$
$$= [\partial_i\pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi\,p_\varphi]\,dt, \tag{67}$$

$$dp_\varphi = -\frac{\partial H_0'}{\partial \varphi}\,dt - \frac{\partial H'}{\partial \varphi}\,dA^0,$$
$$= [(\overrightarrow{D}\cdot\overrightarrow{D}\varphi)^* - m^2\varphi^* - ieA_0 p_\varphi]\,dt, \tag{68}$$

and

$$dp_{\varphi^*} = -\frac{\partial H_0'}{\partial \varphi^*}\,dt - \frac{\partial H'}{\partial \varphi^*}\,dA^0,$$
$$= [(\overrightarrow{D}\cdot\overrightarrow{D}\varphi) - m^2\varphi + ieA_0 p_{\varphi^*}]\,dt. \tag{69}$$

The integrability condition ($dH_\alpha' = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \tag{70}$$

which leads to a new constraint

$$H'' = \partial_i\pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi\,p_\varphi = 0. \tag{71}$$

Taking the total differential of H'', we have

$$dH'' = \partial_i d\pi^i + iep_{\varphi^*} d\varphi^* + ie\varphi^* dp_{\varphi^*} - ie\varphi\, dp_\varphi - iep_\varphi\, d\varphi = 0. \tag{72}$$

Then the set of equation (62-69) is integrable. Therefore, the canonical phase space coordinates (φ, p_φ) and $(\varphi^*, p_{\varphi^*})$ are obtained in terms of parameters (t, A^0).

Making use of Eq. (13) and (59-61), we obtain the canonical action integral as

$$Z = \int d^4x(-\frac{1}{4}\, F^{ij}F_{ij} - \frac{1}{2}\, \pi_i\pi^i + p_\varphi p_{\varphi^*} + \overrightarrow{D}\varphi^* \cdot \overrightarrow{D}\varphi + m^2|\varphi|^2), \tag{73}$$

where

$$\overrightarrow{D} = \overrightarrow{\nabla} + ie\overrightarrow{A}. \tag{74}$$

Now the path integral representation (16) is given by

$$\langle out|S|In\rangle = \int \prod_i dA^i\, d\pi^i\, d\varphi\, dp_\varphi\, d\varphi^*\, dp_{\varphi^*}\, exp\left[i\Big\{ \int d^4x \right.$$
$$\left. (-\frac{1}{2}\, \pi_i\pi^i - \frac{1}{4}\, F^{ij}F_{ij} + p_\varphi p_{\varphi^*} + (D_i\varphi)^*(D_i\varphi) - m^2\varphi^*\varphi)\Big\}\right]. \tag{75}$$

5 Path integral quantization of electromagnetic field coupled to a spinor

We analyse the case of the electromagnetic field coupled to a spinor, whose Hamiltonian formalism was analysed [30,31]. We will consider the Lagrangian density written as

$$L = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi, \tag{76}$$

where A_μ are even variables while ψ and $\overline{\psi}$ are odd ones. The electromagnetic tensor is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and we are adopting the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$.

The Lagrangian function (76) is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}\,, \tag{77}$$

is three.

The momenta variables conjugated, respectively, to A_i, A_0, ψ and $\overline{\psi}$, are

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \tag{78}$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0 = -H_1, \tag{79}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = i\,\overline{\psi}\gamma^0 = -H_\psi\,, \tag{80}$$

$$p_{\overline{\psi}} = \frac{\partial L}{\partial \dot{\overline{\psi}}} = 0 = -H_{\overline{\psi}}\,, \tag{81}$$

where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual, ψ as a column vector and $\overline{\psi}$ as a row vector implies that p_ψ will be a row vector while $p_{\overline{\psi}}$ will be a column vector.

With the aid of relation (78), the Lagrangian density may be written as

$$L = -\frac{1}{2}\,\pi_i\pi^i - \frac{1}{4}\,F_{ij}F^{ij} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi, \tag{82}$$

then the canonical Hamiltonian density takes the form

$$H_0 \;=\; \pi^i\dot{A}_i + \frac{1}{2}\,\pi_i\pi^i + \frac{1}{4}\,F^{ij}F_{ij} - i\overline{\psi}(\gamma^\mu ieA_\mu + \gamma^i\partial_i)\psi + m\overline{\psi}\psi. \tag{83}$$

The velocities $\dot{A}_i$ can be expressed in terms of the momenta π_i as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \tag{84}$$

Therefore, the Hamiltonian density is

$$H_0 \;=\; \frac{1}{4}\,F^{ij}F_{ij} - \frac{1}{2}\,\pi_i\pi^i + \pi^i\partial_i A_0 + \overline{\psi}\gamma^\mu eA_\mu\psi - \overline{\psi}(i\gamma^i\partial_i - m)\psi. \tag{85}$$

The set of Hamilton-Jacobi Partial Differential Equation (HJPDE) reads

$$H_0' = p_0 + \frac{1}{4}\,F^{ij}F_{ij} - \frac{1}{2}\,\pi_i\pi^i + \pi^i\partial_i A_0 + \overline{\psi}\gamma^\mu eA_\mu\psi - \overline{\psi}(i\gamma^i\partial_i - m)\psi, \tag{86}$$

$$H_1' = \pi^0 + H_1 = \pi_0 = 0, \tag{87}$$

$$H_\psi' = p_\psi + H_\psi = p_\psi - i\overline{\psi}\,\gamma^0 = 0, \tag{88}$$

$$H_{\overline{\psi}}' = p_{\overline{\psi}} + H_{\overline{\psi}} = p_{\overline{\psi}} = 0. \tag{89}$$

Therefore, the total differential equations for the characteristic (10), (11) and (12), obtained as

$$\begin{aligned} dA^i &= \frac{\partial H_0'}{\partial \pi_i}\,dt + \frac{\partial H_1'}{\partial \pi_i}\,dA^0 + \frac{\partial H_\psi'}{\partial \pi_i}\,d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \pi_i}\,d\overline{\psi}, \\ &= -(\pi^i + \partial_i A_0)\,dt, \end{aligned} \tag{90}$$

$$\begin{aligned} dA^0 &= \frac{\partial H_0'}{\partial \pi_0}\,dt + \frac{\partial H_1'}{\partial \pi_0}\,dA^0 + \frac{\partial H_\psi'}{\partial \pi_0}\,d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \pi_0}\,d\overline{\psi}, \\ &= dA^0, \end{aligned} \tag{91}$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H_0'}{\partial A_i}\,dt - \frac{\partial H_1'}{\partial A_i}\,dA^0 - \frac{\partial H_\psi'}{\partial A_i}\,d\psi - \frac{\partial H_{\overline{\psi}}'}{\partial A_i}\,d\overline{\psi}, \\ &= (\partial_l F^{li} - e\overline{\psi}\gamma^i\psi)\,dt, \end{aligned} \tag{92}$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H_0'}{\partial A_0}\,dt - \frac{\partial H_1'}{\partial A_0}\,dA^0 - \frac{\partial H_\psi'}{\partial A_0}\,d\psi - \frac{\partial H_{\overline{\psi}}'}{\partial A_0}\,d\overline{\psi}, \\ &= (\partial_i \pi^i - e\overline{\psi}\gamma^0\psi)\,dt, \end{aligned} \tag{93}$$

$$\begin{aligned} dp_\psi &= -\frac{\partial H_0'}{\partial \psi}\,dt - \frac{\partial H_1'}{\partial \psi}\,dA^0 + \frac{\partial H_\psi'}{\partial \psi}\,d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \psi}\,d\overline{\psi}, \\ &= -(i\gamma^i\partial_i + e\gamma^\mu A_\mu + m)\overline{\psi}\,dt, \end{aligned} \tag{94}$$

and

$$\begin{aligned} dp_{\overline{\psi}} &= -\frac{\partial H_0'}{\partial \overline{\psi}}\,dt - \frac{\partial H_1'}{\partial \overline{\psi}}\,dA^0 + \frac{\partial H_\psi'}{\partial \overline{\psi}}\,d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \overline{\psi}}\,d\overline{\psi}, \\ &= (-i\gamma^i\partial_i + e\gamma^\mu A_\mu + m)\psi\,dt - i\gamma^0 d\psi. \end{aligned} \tag{95}$$

The integration condition ($dH'_\alpha = 0$) imply that the variation of the constraints H'_1, H'_ψ and $H'_{\overline{\psi}}$ should be identically zero

$$dH'_1 = d\pi_0 = 0, \tag{96}$$

$$dH'_\psi = dp_\psi - i\gamma^0 d\overline{\psi} = 0, \tag{97}$$

$$dH'_{\overline{\psi}} = dp_{\overline{\psi}} = 0, \tag{98}$$

when we substituting from Eqs. (94) and (95) into Eqs.(97) and (98) respectively, we obtained as

$$dH'_\psi = 0, \tag{99}$$

and

$$dH'_{\overline{\psi}} = 0, \tag{100}$$

if and only if

$$i\overline{\psi}\gamma^\mu(\overleftarrow{\partial}_\mu - ieA_\mu) + m\overline{\psi} = 0, \tag{101}$$

and

$$i(\partial_\mu + ieA_\mu)\gamma^\mu\psi - m\psi = 0, \tag{102}$$

are satisfied. Then the set of equations (90, 92, 93) are integrable and are just ordinary differential equations and are set in the form

$$\dot{A}^i = -\pi^i - \partial_i A_0, \tag{103}$$

$$\dot{\pi}^i = \partial_l F^{li} - e\overline{\psi}\gamma^i\psi, \tag{104}$$

$$\dot{\pi}^0 = \partial_i \pi^i - e\overline{\psi}\gamma^0\psi. \tag{105}$$

These are the equations of motions with full gauge freedom. It can be seen, from Eq. (91), that A^0 is an arbitrary (gauge dependent) variable since its time derivative is arbitrary. Besides that, Eq. (103) shows the gauge dependence of A^i and, Taking the curl of its vector form, leads to the known Maxwell equation

$$\frac{\partial\overrightarrow{A}}{\partial t} = -\overrightarrow{E} - \overrightarrow{\nabla}(A_0 - \alpha) \Rightarrow \frac{\partial\overrightarrow{B}}{\partial t} = -\overrightarrow{\nabla} \times \overrightarrow{E}. \tag{106}$$

Writing $j^{\mu} = e\overline{\psi}\gamma^{\mu}\psi$ we get, from Eq. (104), the inhomogeneous Maxwell equation

$$\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \vec{j}, \tag{107}$$

while the other inhomogeneous equation

$$\vec{\nabla} \cdot \vec{E} = j^{0}, \tag{108}$$

follows from Eq. (105). Expressions (101) and (102) are the known equations for the spinor ψ and $\overline{\psi}$.

Eqs. (12) and (86-89) lead us to the canonical action integral as

$$Z = \int d^4x \Big(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i \pi^i + \pi^i \dot{A}_i + \pi^i \partial_i A_0 + i\overline{\psi}\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi - m\overline{\psi}\psi \Big). \tag{109}$$

Making use of equations (14) and (109), we obtained the path integral as

$$\langle out|S|In\rangle = \int \prod_i dA^i \, d\pi^i \, d\psi \, d\overline{\psi} \, exp\Big[i\Big\{ \int d^4x \Big(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i \pi^i + \pi^i \dot{A}_i + \pi^i \partial_i A_0 + i\overline{\psi}\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi - m\overline{\psi}\psi \Big) \Big\} \Big]. \tag{110}$$

Integration over π_i gives

$$\langle out|S|In\rangle = N \int \prod_i dA^i \, d\psi \, d\overline{\psi} \, exp\Big[i\Big\{ \int d^4x \Big(\frac{1}{2} (\dot{A}^i + \partial_i A_0)^2 - \frac{1}{4} F^{ij} F_{ij} + i\overline{\psi}\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi - m\overline{\psi}\psi \Big) \Big\} \Big]. \tag{111}$$

6 Conclusion

In this paper three different constrained fields systems are studied by using Hamilton-Jacobi formulation. Firstly, he scalar field coupled to two flavours of fermions through Yukawa couplings is discussed as constrained

system by using Hamilton-Jacobi methods. The equations of motion are obtained without introducing Lagrange multipliers to the canonical Hamiltonian.

Then, we obtained the path integral quantization of the scalar field coupled minimally to the vector potential by using the canonical path integral formulation. The integrability conditions dH_0' and dH' are satisfied, the system is integrable, hence the path integral is obtained directly as an integration over the canonical phase space coordinates$A^i, \pi^i, \varphi, P_\varphi, \varphi^*$ and p_{φ^*} without using any gauge fixing conditions.

Finally, the path integral quantization of the electromagnetic field coupled to a spinor is also obtained by using the canonical path integral formalism. The integrability conditions dH_1', dH_ψ' and $dH_{\overline{\psi}}'$ are identically satisfied, and the system is integrable. Hence, the canonical phase space coordinates (A^i, π^i), (ψ, p_ψ) and $(\overline{\psi}, p_{\overline{\psi}})$ are obtained in terms of the parameter τ. The path integral is obtained as an integration over the canonical phase-space coordinates (A^i, π^i) and $(\psi, \overline{\psi})$ without using any gauge fixing condition. From the equations of motion for this system, we obtained the inhomogeneous Maxwell equation.

One can see many advantages of this path integral formalism, which are no need to enlarge the initial phase-space by introducing unphysical auxiliary field, no need to distinguish between first and second-class constraints, no need to introduce Lagrange multiplierts, no need to use delta functions in the measure as well as to use gauge fixing conditions; all that needed is the set of Hamilton-jacobi partial differential equations and the equations of motions. If the system is integrable, then one can construct the reduced canonical phase-space.

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A KIND OF GEOMETRY INEQUALITIES INVOLVING AN INTERIOR POINT OF A TRIANGLE

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Abstract

In this paper, we establish three geometric inequalities involving an arbitrary point of a triangle. We also propose three related conjectures which have been checked by the computer.

Key words: triangle, interior point, medians, Euler's inequality, Gerretsen's inequality, the fundamental triangle inequality

1. Introduce and main results

In a Chinese paper [1], the author established the following geometric inequality:

Let P be an interior point of a triangle ABC. Let R_1, R_2, R_3 be the distances from P to the vertices A, B, C, respectively, and let r_1, r_2, r_3 be the distances from P to the sides BC, CA, AB, respectively. Denote by a, b, c the side lengths BC, CA, AB, respectively. Then for any real numbers x, y, z the following inequality holds:

$$a\frac{R_1}{r_1}x^2 + b\frac{R_2}{r_2}y^2 + c\frac{R_3}{r_3}z^2 \geq 2(yza + zxb + xyc), \tag{1.1}$$

with equality if and only if $x : y : z = \cos A : \cos B : \cos C$ and P is the circumcenter of triangle ABC.

In the Chapter III of the author's monograph [2], several equivalent forms of inequality (1.1) are given. One of them is

$$aR_1x^2 + bR_2y^2 + cR_3z^2 \geq 2(yzar_1 + zxbr_2 + xycr_3), \tag{1.2}$$

with equality if and only if $x = y = z$ and P is the orthocenter of triangle ABC. We also gave some applications of inequalities (1.1) and (1.2) in [1] and [2].

An obvious corollary of (1.3) is

$$a\frac{R_1}{r_1} + b\frac{R_2}{r_2} + c\frac{R_3}{r_3} \geq 4s, \tag{1.3}$$

where s is the semi-perimeter of triangle ABC. The following two similar inequalities were also obtained from (1.1):

$$(s-a)\frac{R_1}{r_1} + (s-b)\frac{R_2}{r_2} + (s-c)\frac{R_3}{r_3} \geq 2s, \tag{1.4}$$

$$(b+c)\frac{R_1}{r_1} + (c+a)\frac{R_2}{r_2} + (a+b)\frac{R_3}{r_3} \geq 4s. \tag{1.5}$$

Recently, inspired by inequality (1.3), the author first found that the following similar inequality holds:

$$m_a\frac{R_1}{r_1} + m_b\frac{R_2}{r_2} + m_c\frac{R_3}{r_3} \geq 2(m_a + m_b + m_c). \tag{1.6}$$

where m_a, m_b, m_c are the corresponding medians of triangle ABC. Subsequently, we further obtain the following stronger result:

Theorem 1. *For an interior point P of triangle ABC the following inequality holds:*

$$m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}\geq\sqrt{12(m_a^2+m_b^2+m_c^2)}, \tag{1.7}$$

with equality if and only if triangle ABC is equilateral and P is its center.

By the power mean inequality, we know that inequality (1.7) is stronger than (1.6). In addition, by the well-known identity:

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2), \tag{1.8}$$

we see that inequality (1.7) is equivalent to

$$m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}\geq 3\sqrt{a^2+b^2+c^2}. \tag{1.9}$$

For an acute triangle ABC, we have the following inequality (see [3, p.248])

$$a^2+b^2+c^2\geq 4(R+r)^2, \tag{1.10}$$

where R and r are the circumradius and inradius of triangle ABC. This inequality and inequality (1.8) yields that the following inequality (1.11) holds for the acute triangle. We shall further prove that it holds for any triangle.

Theorem 2. *For an interior point P of triangle ABC the following inequality holds:*

$$m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}\geq 6(R+r), \tag{1.11}$$

with equality if and only if triangle ABC is equilateral and P is its center.

Motivated by inequality (1.6), we shall prove the following similar result:

Theorem 3. *For an interior point P of triangle ABC the following inequality holds:*

$$m_a^2\frac{R_1}{r_1}+m_b^2\frac{R_2}{r_2}+m_c^2\frac{R_3}{r_3}\geq 2(m_a^2+m_b^2+m_c^2), \tag{1.12}$$

with equality if and only if triangle ABC is equilateral and P is its center.

Identity (1.8) shows that inequality (1.12) is equivalent to

$$m_a^2\frac{R_1}{r_1}+m_b^2\frac{R_2}{r_2}+m_c^2\frac{R_3}{r_3}\geq\frac{3}{2}(a^2+b^2+c^2). \tag{1.13}$$

The main purpose of this paper is to prove Theorem 1, 2 and 3. We also propose three related conjectures.

2. Proof of Theorem 1

In order to prove Theorem 1, we need several lemmas. In what follows, we shall continuously use the above symbols. In addition, we shall denote $\sum$ by cyclic sums over a triple or multiple triples. For example,

$$\begin{aligned}&\sum b^2c^2=b^2c^2+c^2a^2+a^2b^2,\\&\sum m_a^2=m_a^2+m_b^2+m_c^2,\\&\sum m_a\frac{R_1}{r_1}=m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}.\end{aligned}$$

Lemma 1. *For an interior point of triangle ABC the following inequality holds:*

$$2R_1\sin\frac{A}{2}\geq r_2+r_3, \tag{2.1}$$

with equality if and only if $r_2=r_3$.

Inequality (2.1) is easily proved (see the proof of inequality 12.20 in [4]). Two similar relations are also valid.

Lemma 2. *For any positive real numbers x,y,z and u,v,w, the following inequality holds:*

$$\sum\frac{x^3}{u^2}\geq\frac{\left(\sum x\right)^3}{\left(\sum u\right)^2}, \tag{2.2}$$

with equality if and only if $x:y:z=u:v:w$.

Inequality (2.2) is a special case of the Radon inequality (cf. [5, Theorem 65]).

Lemma 3. *In any triangle ABC the following inequality holds:*

$$\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{s-a}{2a}\left(\frac{s-b}{c}+\frac{s-c}{b}\right), \tag{2.3}$$

with equality if and only if $b=c$.

Proof

Using the following well-known formula in triangle ABC:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \tag{2.4}$$

we get

$$\sin\frac{B}{2}\sin\frac{C}{2} = \frac{s-a}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}. \tag{2.5}$$

Then inequality (2.3) follows immediately by using the simplest arithmetic-geometric mean inequality. Also, the equality occurs only when $(s-b)/c=(s-c)/b$, i.e., $b=c$. Lemma 3 is proved.

Lemma 4. *In any triangle ABC the following inequality holds:*

$$m_a \le \frac{8S^2+bc(b-c)^2}{4aS}, \tag{2.6}$$

where S is the area of triangle ABC. Equality in (2.6) *holds if and only if* $b=c$.

Inequality (2.6) is one of the equivalent conclusion of Theorem 1.1 from [6].

Lemma 5. *In any triangle ABC the following inequality holds:*

$$m_a m_b m_c \ge \frac{1}{8R}\sum b^2c^2, \tag{2.7}$$

with equality if and only if triangle ABC is isosceles.

Inequality (2.7) and its equivalent forms have been used by the author in [7] and [8].

The following two lemmas give some identities for the sums $\sum a^n$ and $\sum (bc)^n$ in term of R, r and s.

Lemma 6. *In the triangle ABC, let* $p_n = \sum a^n$ *(n being natural number). Then the following identities hold:*

$$p_1 = 2s, \tag{2.8}$$

$$p_2 = 2s^2 - 8Rr - 2r^2, \tag{2.9}$$

$$p_3 = 2s^3 - (12Rr + 6r^2)s, \tag{2.10}$$

$$p_4 = 2s^4 - 4(4R + 3r)rs^2 + 2(4R + r)^2r^2, \tag{2.11}$$

$$p_5 = 2s^5 - 20(R + r)rs^3 + 10(2R + r)(4R + r)r^2s, \tag{2.12}$$

$$\begin{aligned} p_6 = {} & 2s^6 - 6(4R + 5r)rs^4 + 6(24R^2 + 24Rr + 5r^2)r^2s^2 \\ & - 2(4R + r)^3r^3, \end{aligned} \tag{2.13}$$

$$\begin{aligned} p_7 = {} & 2s^7 - 14(2R + 3r)rs^5 + 14(16R^2 + 20Rr + 5r^2)r^2s^3 \\ & - 14(2R + r)(4R + r)^2r^3s, \end{aligned} \tag{2.14}$$

$$\begin{aligned} p_8 = {} & 2s^8 - 8(4R + 7r)rs^6 + 20(16R^2 + 24Rr + 7r^2)r^2s^4 \\ & - 8(4R + r)(32R^2 + 32Rr + 7r^2)r^3s^2 + 2(4R + r)^4r^4, \end{aligned} \tag{2.15}$$

$$\begin{aligned} p_9 = {} & 2s^9 - 36(R + 2r)rs^7 + 36(12R^2 + 21Rr + 7r^2)r^2s^5 \\ & - 12(160R^3 + 240R^2r + 105Rr^2 + 14r^3)r^3s^3 \\ & + 18(2R + r)(4R + r)^3r^4s, \end{aligned} \tag{2.16}$$

$$\begin{aligned} p_{10} = {} & 2s^{10} - 10(4R + 9r)rs^8 + 140(2R + 3r)(2R + r)r^2s^6 \\ & - 20(160R^3 + 280R^2r + 140Rr^2 + 21r^3)r^3s^4 \\ & + 10(40R^2 + 40Rr + 9r^2)(4R + r)^2r^4s^2 - 2(4R + r)^5r^5, \end{aligned} \tag{2.17}$$

$$\begin{aligned} p_{11} = {} & 2s^{11} - 22(2R + 5r)rs^9 + 44(16R^2 + 36Rr + 15r^2)r^2s^7 \\ & - 308(2R + r)(8R^2 + 12Rr + 3r^2)r^3s^5 \\ & + 22(4R + r)(160R^3 + 240R^2r + 108Rr^2 + 15r^3)r^4s^3 \\ & - 22(2R + r)(4R + r)^4r^5s, \end{aligned} \tag{2.18}$$

$$\begin{aligned} p_{12} = {} & 2s^{12} - 12(4R + 11r)rs^{10} + 18(48R^2 + 120Rr + 55r^2)r^2s^8 \\ & - 56(128R^3 + 288R^2r + 180Rr^2 + 33r^3)r^3s^6 + 6(4480R^4 \\ & + 8960R^3r + 6048R^2r^2 + 1680Rr^3 + 165r^4)r^4s^4 \\ & - 12(48R^2 + 48Rr + 11r^2)(4R + r)^3r^5s^2 + 2(4R + r)^6r^6, \end{aligned} \tag{2.19}$$

$$
\begin{aligned}
p_{13} =& 2s^{13} - 52(R+3r)rs^{11} + 130(8R^2+22Rr+11r^2)r^2s^9 \\
& - 312(32R^3+80R^2r+55Rr^2+11r^3)r^3s^7 + 26(1792R^4 \\
& + 4032R^3r + 3024R^2r^2 + 924Rr^3 + 99r^4)r^4s^5 \\
& - 52(112R^3+168R^2r+77Rr^2+11r^3)(4R+r)^2r^5s^3 \\
& + 26(2R+r)(4R+r)^5r^6s, \qquad (2.20) \\
p_{14} =& 2s^{14} - 14(4R+13r)rs^{12} + 154(8R^2+24Rr+13r^2)r^2s^{10} \\
& - 42(320R^3+880R^2r+660Rr^2+143r^3)r^3s^8 \\
& + 42(1792R^4+4480R^3r+3696R^2r^2+1232Rr^3 \\
& + 143r^4)r^4s^6 - 14(4R+r)(3584R^4+7168R^3r \\
& + 4928R^2r^2+1408Rr^3+143r^4)r^5s^4 \\
& + 14(56R^2+56Rr+13r^2)(4R+r)^4r^6s^2 \\
& - 2(4R+r)^7r^7, \qquad (2.21) \\
p_{16} =& 2s^{16} - 16(4R+15r)rs^{14} + 104(16R^2+56Rr+35r^2)r^2s^{12} \\
& - 176(128R^3+416R^2r+364Rr^2+91r^3)r^3s^{10} \\
& + 132(1280R^4+3840R^3r+3744R^2r^2+1456Rr^3 \\
& + 195r^4)r^4s^8 - 16(43008R^5+118272R^4r+118272R^3r^2 \\
& + 54912R^2r^3+12012Rr^4+1001r^5)r^5s^6 + 8(10752R^4 \\
& + 21504R^3r+14976R^2r^2+4368Rr^3+455r^4)(4R+r)^2r^6s^4 \\
& - 16(8R+5r)(8R+3r)(4R+r)^5r^7s^2 \\
& + 2(4R+r)^8r^8. \qquad (2.22)
\end{aligned}
$$

Lemma 7. *In the triangle ABC, let $q_n = \sum(bc)^n$ (n being natural number). Then the following identities hold:*

$$
\begin{aligned}
q_1 =& s^2 + 4Rr + r^2, \qquad (2.23) \\
q_2 =& s^4 - 2(4R-r)rs^2 + (4R+r)^2r^2, \qquad (2.24) \\
q_3 =& s^6 - 3(4R-r)rs^4 + 3r^4s^2 + (4R+r)^3r^3, \qquad (2.25) \\
q_5 =& s^{10} - 5(4R-r)rs^8 + 10(8R^2-4Rr+r^2)r^2s^6 \\
& + 10r^6s^4 + 5(4R+r)^2r^6s^2 + (4R+r)^5r^5, \qquad (2.26)
\end{aligned}
$$

$$\begin{aligned}
q_6 =& s^{12} - 6(4R-r)rs^{10} + 3(48R^2 - 24Rr + 5r^2)r^2s^8 \\
&- 4(32R^3 - 24R^2r + 12Rr^2 - 5r^3)r^3s^6 \\
&+ 3(16R+5r)r^7s^4 + 6(4R+r)^3r^7s^2 \\
&+ (4R+r)^6r^6, \qquad (2.27)\\
q_8 =& s^{16} - 8(4R-r)rs^{14} + 4(80R^2 - 40Rr + 7r^2)r^2s^{12} \\
&- 8(128R^3 - 96R^2r + 36Rr^2 - 7r^3)r^3s^{10} \\
&+ 2(256R^4 - 256R^3r + 160R^2r^2 - 80Rr^3 + 35r^4)r^4s^8 \\
&+ 8(20R+7r)r^9s^6 + 4(16R+7r)(4R+r)^2r^9s^4 \\
&+ 8(4R+r)^5r^9s^2 + (4R+r)^8r^8. \qquad (2.28)
\end{aligned}$$

Remark 1. *Identities* (2.9), (2.10), (2.11) *and* (2.23) *in Lemma 6 and 7 are well known. The proofs of the other identities can be found in [9] and [10]. Identities* (2.20)-(2.24),(2.26) *and* (2.28) *will not be used in the proof of Theorem 1, but will be used in the proof of Theorem 2.*

We are now ready to prove Theorem 1.

Proof

By Lemma 1, to prove inequality (1.7) we need to show that

$$\sum m_a \frac{r_2 + r_3}{2r_1 \sin\dfrac{A}{2}} \geq \sqrt{12\sum m_a^2},$$

which is equivalent to

$$\sum \left(\frac{m_b r_3}{r_2 \sin\dfrac{B}{2}} + \frac{m_c r_2}{r_3 \sin\dfrac{C}{2}} \right) \geq 2\sqrt{12\sum m_a^2}. \qquad (2.29)$$

Making use of the simplest arithmetic-geometric mean inequality, we have

$$\frac{m_b r_3}{r_2 \sin\dfrac{B}{2}} + \frac{m_c r_2}{r_3 \sin\dfrac{C}{2}} \geq 2\sqrt{\frac{m_b m_c}{\sin\dfrac{B}{2}\sin\dfrac{C}{2}}}.$$

Thus, note that the previous identity (1.8), we only need to prove

$$\sum\sqrt{\frac{m_bm_c}{\sin\frac{B}{2}\sin\frac{C}{2}}} \geq 3\sqrt{\sum a^2}. \tag{2.30}$$

Denoting by r_a, r_b, r_c the radii of escribed circle of triangle ABC. In Lemma 2, we set $x = r_a, y = r_b, z = r_c$ and

$$u = \sqrt{\frac{m_bm_c}{\sin\frac{B}{2}\sin\frac{C}{2}}}, v = \sqrt{\frac{m_cm_a}{\sin\frac{C}{2}\sin\frac{A}{2}}}, w = \sqrt{\frac{m_am_b}{\sin\frac{A}{2}\sin\frac{B}{2}}},$$

then it follows that

$$\left(\sum\sqrt{\frac{m_bm_c}{\sin\frac{B}{2}\sin\frac{C}{2}}}\right)^2 \sum\frac{r_a^3}{m_bm_c}\sin\frac{B}{2}\sin\frac{C}{2} \geq \left(\sum r_a\right)^3. \tag{2.31}$$

Consequently, to prove inequality (2.30) it remains to show that

$$\left(\sum r_a\right)^3 \geq 9\sum\frac{r_a^3}{m_bm_c}\sin\frac{B}{2}\sin\frac{C}{2}\sum a^2. \tag{2.32}$$

Since

$$r_a = \frac{S}{s-a}, \tag{2.33}$$

using Lemma 3-5, we have

$$\begin{aligned}
&\sum\frac{r_a^3}{m_bm_c}\sin\frac{B}{2}\sin\frac{C}{2}\\
&= \frac{S^3}{m_am_bm_c}\sum\frac{m_a}{(s-a)^3}\sin\frac{B}{2}\sin\frac{C}{2}\\
&\leq \frac{8RS^3}{\sum b^2c^2}\sum\frac{1}{(s-a)^3}\cdot\frac{8S^2+bc(b-c)^2}{4aS}\cdot\frac{s-a}{2a}\left(\frac{s-b}{c}+\frac{s-c}{b}\right)\\
&= \frac{RS^2}{\sum b^2c^2}\sum\frac{8S^2+bc(b-c)^2}{a^2(s-a)^2}\left(\frac{s-b}{c}+\frac{s-c}{b}\right).
\end{aligned}$$

Thus, to prove inequality (2.32) we need to show that

$$\left(\sum r_a\right)^3 \geq \frac{9RS^2}{\sum b^2c^2} \sum \frac{8S^2 + bc(b-c)^2}{a^2(s-a)^2} \left(\frac{s-b}{c} + \frac{s-c}{b}\right) \sum a^2. \tag{2.34}$$

Using the known identity:

$$abc = 4SR, \tag{2.35}$$

we easily know that (2.34) is equivalent to

$$\begin{aligned} &16R\left(\sum r_a\right)^3 \sum b^2c^2 \prod (s-a)^2 \\ &\geq 9\sum a^2 \sum bc(s-b)^2(s-c)^2[8S^2 + bc(b-c)^2][(s-b)b + (s-c)c], \end{aligned} \tag{2.36}$$

where $\prod$ denotes cyclic product. Again, using the following known identities

$$\sum r_a = 4R + r, \tag{2.37}$$

$$\prod (s-a) = sr^2, \tag{2.38}$$

one sees that inequality (2.36) is equivalent to

$$A_0 \equiv 16R(4R+r)^3 r^4 s^2 \sum b^2c^2 - 9B_0 \sum a^2 \geq 0, \tag{2.39}$$

where

$$B_0 = \sum bc(s-b)^2(s-c)^2[8S^2 + bc(b-c)^2][(s-b)b + (s-c)c].$$

Next, we shall compute B_0 in terms of R, r and s. With the help of MAPLE software, using $s = (a+b+c)/2$ and Heron's formula

$$S = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.40}$$

one can easily obtain the following identity:

$$
\begin{aligned}
64B_0 = & -5b^8ca^3 + 8b^6c^3a^3 - 104b^5c^5a^2 + 90b^4c^6a^2 + 8b^3c^6a^3 \\
& - 76b^3c^7a^2 + 8b^3c^3a^6 + 90b^2c^6a^4 + 46b^2c^8a^2 + 90b^4c^2a^6 \\
& - 180b^4c^4a^4 + 44b^3c^4a^5 - 76b^2c^7a^3 + 44b^4c^5a^3 - 76b^7c^3a^2 \\
& + 29b^9c^3 - 64b^8c^4 + 98b^7c^5 - 112b^6c^6 + 98b^5c^7 - 8b^2c^{10} \\
& - 64b^4c^8 - 8b^{10}c^2 + b^{11}c + 44b^4c^3a^5 + 90b^2c^4a^6 - 5b^3c^8a \\
& + 46b^2c^2a^8 + 44b^3c^5a^4 + 20b^7c^4a - 104b^2c^5a^5 + 44b^5c^3a^4 \\
& + 20b^4c^7a + 90b^6c^2a^4 + bc^{11} + 20b^4ca^7 - 14b^5ca^6 \\
& - 14b^6c^5a - 5b^8c^3a - 104b^5c^2a^5 - 2b^{10}ca - 14b^5c^6a \\
& - 14bc^5a^6 - 5bc^8a^3 - 14b^6ca^5 + 20b^7ca^4 - 76b^7c^2a^3 \\
& + 44b^5c^4a^3 - 14bc^6a^5 + 46b^8c^2a^2 - 2bc^{10}a + 90b^6c^4a^2 \\
& + 29b^3c^9 - 8c^2a^{10} + 29c^3a^9 + 20bc^7a^4 + 20bc^4a^7 \\
& - 5b^3ca^8 - 5bc^3a^8 - 64c^8a^4 + 29c^9a^3 + ca^{11} - 8c^{10}a^2 \\
& - 112c^6a^6 + 98c^7a^5 - 64c^4a^8 + 98c^5a^7 - 76c^2a^7b^3 \\
& - 76c^3a^7b^2 + 29a^9b^3 + c^{11}a - 64a^8b^4 + 98a^7b^5 \\
& - 2ca^{10}b - 8a^{10}b^2 + a^{11}b - 64a^4b^8 - 112a^6b^6 \\
& + 98a^5b^7 + 29a^3b^9 - 8a^2b^{10} + ab^{11}.
\end{aligned} \tag{2.41}
$$

We set $d = abc$, then it is not difficult to obtain the following identity:

$$
\begin{aligned}
64B_0 = & -180d^4 + (44p_1p_2 - 36p_3)d^3 + (32p_6 - 76p_1p_5 + 90p_2p_4 \\
& - 104q_3)d^2 + (20p_3p_6 - 3p_9 - 14p_4p_5 - 5p_2p_7)d - 56p_{12} \\
& + 98p_5p_7 + 29p_3p_9 - 112q_6 - 64p_4p_8 - 8p_2p_{10} + p_1p_{11}.
\end{aligned} \tag{2.42}
$$

With the help of MAPLE software, using some related identities given in Lemma 6, 7 and the following identity:

$$abc = 4Rrs \tag{2.43}$$

one can easily obtain the following identity:

$$
\begin{aligned}
B_0 = & 8r^5 \left[(-40R^3 - 100R^2r - 14Rr^2 + r^3)s^4 \right. \\
& \left. + 2(2R + 11r)(4R + r)^3Rs^2 - (4R + r)^6r\right].
\end{aligned} \tag{2.44}
$$

Now, using identities (2.10),(2.25) and (2.44), we easily get

$$A_0 = 16r^4C_0, \tag{2.45}$$

where

$$\begin{aligned}C_0 =&(64R^4 + 408R^3r + 912R^2r^2 + 127Rr^3 - 9r^4)s^6\\ &- (4R + r)(704R^4 + 3848R^3r + 2512R^2r^2 + 322Rr^3\\ &- 9r^4)rs^4 + (184R^2 + 271Rr + 9r^2)(4R + r)^4r^2s^2\\ &- 9(4R + r)^7r^3.\end{aligned}$$

Consequently, for proving inequality (2.39) we have to prove that

$$C_0 \geq 0. \tag{2.46}$$

We recall that for any triangle we have Euler's inequality

$$e \equiv R - 2r \geq 0 \tag{2.47}$$

and Gerretsen's inequality (cf. [3], [4] and [11]):

$$x_0 \equiv s^2 - 16Rr + 5r^2 \geq 0. \tag{2.48}$$

Based on these two inequality, after analysis we obtain the following identity:

$$C_0 = C_1x_0^3 + rC_2x_0^2 + 4r^2C_3, \tag{2.49}$$

where

$$\begin{aligned}C_1 =&64R^4 + 408R^3r + 912R^2r^2 + 127Rr^3 - 9r^4,\\ C_2 =&256R^5 + 2528R^4r + 23760R^3r^2 - 11384R^2r^3\\ &- 2623Rr^4 + 144r^5,\\ C_3 =&(1536R^6 - 21952R^5r + 78752R^4r^2 - 65250R^3r^3\\ &+ 8690R^2r^4 + 4352Rr^5 - 189r^6)s^2 + (12288R^7\\ &+ 140288R^6r - 908096R^5r^2 + 1017944R^4r^3\\ &- 311314R^3r^4 - 25502R^2r^5 + 15422Rr^6 - 621r^7)r.\end{aligned}$$

By Euler's inequality one sees that $C_1 > 0$ and $C_2 > 0$. Since $x_0 \geq 0$, to prove $C_0 \geq 0$ it remains to show $C_3 \geq 0$.

For any triangle we have the following two inequalities (due to Yang, cf. [11]):

$$i_0 \equiv (R-r)s^2 - r(16R^2 - 20Rr + 3r^2) \geq 0, \tag{2.50}$$

$$j_0 \equiv 4R^3 - 2Rr^2 - r^3 - (R-r)s^2 \geq 0, \tag{2.51}$$

which are stronger than Gerretsen's inequalities (2.48) and (3.22) below respectively. We easily check the following identity:

$$(e+r)C_3 = C_4 i_0 + C_5 j_0 + C_6, \tag{2.52}$$

where

$$\begin{aligned}
C_4 =& 1536e^6 + 119718e^2r^4 + 314928er^5 + 177147r^6,\\
C_5 =& 2re^3(1760e^2 + 24304er + 33777r^2),\\
C_6 =& er(22784e^7 + 113280e^6r + 562616e^5r^2 + 1650152e^4r^3\\
&+ 2126250e^3r^4 + 1134486e^2r^5 + 284310er^6 + 59049r^7).
\end{aligned}$$

By Euler's inequality $e \geq 0$ we have $C_4 > 0, C_5 \geq 0$ and $C_6 \geq 0$. And then from (2.52) we deduce $C_3 \geq 0$. Inequalities (2.39), (2.30) and (1.7) are proved. Moreover, it is easy to determine that equality in (1.7) holds only when triangle ABC is equilateral and P is its center. This completes the proof of Theorem 1.

3. Proof of Theorem 2

In this section, we shall prove Theorem 2. We first give two lemmas.

Lemma 8. *In any triangle ABC the following inequality holds:*

$$\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{s-a}{b+c}, \tag{3.1}$$

with equality if and only if $b = c$.

Proof

From the previous identity (2.5), using formula (2.4) and the well-known inequality:

$$\sin\frac{A}{2} \leq \frac{a}{b+c}, \tag{3.2}$$

we immediately obtain inequality (3.1). Also, the equality condition of (3.1) is the same as (3.2), i.e., $b = c$. The proof is completed.

Remark 2. *It is easy to prove that inequality* (3.1) *is weaker than the previous inequality* (2.3).

Lemma 9. *In any triangle ABC the following identity holds:*

$$16(m_a m_b m_c)^2$$
$$= s^6 - 3(4R - 11r)rs^4 - 3(20R^2 + 40Rr + 11r^2)r^2s^2 - (4R + r)^3r^3. \tag{3.3}$$

Proof

Using the known median formula

$$4m_a^2 = 2b^2 + 2c^2 - a^2 \tag{3.4}$$

we easily get

$$64(m_a m_b m_c)^2 = 6\sum a^2 \sum a^4 - 10\sum a^6 + 3(abc)^2. \tag{3.5}$$

Then identity (3.3) follows immediately by using the previous identities (2.9), (2.11), (2.13) and (2.43). The poof is completed.

We now prove Theorem 2.

Proof

By Lemma 1, to prove inequality (1.11) we need to show

$$\sum m_a \frac{r_2 + r_3}{r_1 \sin\frac{A}{2}} \geq 12(R + r),$$

which is equivalent to

$$\sum \left(\frac{r_3 m_b}{r_2 \sin\frac{B}{2}} + \frac{r_2 m_c}{r_3 \sin\frac{C}{2}} \right) \geq 12(R + r). \tag{3.6}$$

By the arithmetic-geometric mean inequality, we only need to prove that

$$\sum \sqrt{\frac{m_b m_c}{\sin\frac{B}{2}\sin\frac{C}{2}}} \geq 6(R + r). \tag{3.7}$$

Denoting by h_a, h_b, h_c by the corresponding altitudes of triangle ABC and denoting by r_a, r_b, r_c the radii of escribed circle of triangle ABC. In inequality given in Lemma 2, we take

$$x = h_a + r_a,\ y = h_b + r_b,\ z = h_c + r_c,$$

$$u = \sqrt{\frac{m_b m_c}{\sin\frac{B}{2}\sin\frac{C}{2}}},\ v = \sqrt{\frac{m_c m_a}{\sin\frac{C}{2}\sin\frac{A}{2}}},\ w = \sqrt{\frac{m_a m_b}{\sin\frac{A}{2}\sin\frac{B}{2}}}.$$

Then it follows that

$$\left(\sum\sqrt{\frac{m_b m_c}{\sin\frac{B}{2}\sin\frac{C}{2}}}\right)^2 \sum\frac{(h_a+r_a)^3}{m_b m_c}\sin\frac{B}{2}\sin\frac{C}{2} \ge \left[\sum(h_a+r_a)\right]^3. \tag{3.8}$$

Thus, to prove inequality (3. 7) we only need to show

$$\left[\sum(h_a+r_a)\right]^3 \ge 36(R+r)^2\sum\frac{(h_a+r_a)^3}{m_b m_c}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{3.9}$$

Using $h_a = 2S/a$ and $r_a = S/(s-a)$ we get

$$h_a + r_a = \frac{(b+c)S}{a(s-a)}. \tag{3.10}$$

And, then by Lemma 8 we have

$$(h_a+r_a)^3\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{(b+c)^2 S^3}{a^3(s-a)^2}. \tag{3. 11}$$

Therefore, to prove inequality (3. 9) it remains to prove that

$$\left(\sum h_a + \sum r_a\right)^3 \ge 36(R+r)^2 S^3\sum\frac{(b+c)^2}{m_b m_c a^3(s-a)^2}. \tag{3. 12}$$

Note that

$$2m_b m_c \le \frac{c(s-c)}{b(s-b)}m_b^2 + \frac{b(s-b)}{c(s-c)}m_c^2. \tag{3. 13}$$

To prove inequality (3.12) we need to show that

$$\left(\sum h_a + \sum r_a\right)^3$$
$$\geq 18(R+r)^2 S^3 \sum \left[\frac{c(s-c)}{b(s-b)} m_b^2 + \frac{b(s-b)}{c(s-c)} m_c^2\right] \frac{(b+c)^2}{m_b^2 m_c^2 a^3 (s-a)^2}.$$

Multiplying both sides by $8(m_a m_b m_c)^2 R^3$, and using $2Rh_a = bc$ and identity $abc = 4SR$, one can obtain the following equivalent inequality:

$$(m_a m_b m_c)^2 \left(\sum bc + 8R^2 + 2Rr\right)^3$$
$$\geq 144(R+r)^2 R^3 S^3 \sum \frac{(b+c)^2 m_a^2}{a^3(s-a)^2} \left[\frac{c(s-c)}{b(s-b)} m_b^2 + \frac{b(s-b)}{c(s-c)} m_c^2\right],$$

Again, multiplying both sides by $16 \prod (s-a)^2$ and using identity $abc = 4SR$, the above inequality becomes

$$16(m_a m_b m_c)^2 \left(\sum bc + 8R^2 + 2Rr\right)^3 \prod (s-a)^2$$
$$\geq 36(R+r)^2$$
$$\cdot \sum (s-b)(s-c)(b+c)^2 b^2 c^2 m_a^2 \left[b^2(s-b)^2 m_c^2 + c^2(s-c)^2 m_b^2\right]. \quad (3.14)$$

Using the previous identities (2.23) and (2.38), we further know that the above inequality is equivalent to

$$D_0 \equiv 16r^2 s^4 \left(s^2 + 8R^2 + 6Rr + r^2\right)^3 (m_a m_b m_c)^2 - 36(R+r)^2 E_0 \geq 0, \quad (3.15)$$

where

$$E_0 = \sum (s-b)(s-c)(b+c)^2 b^2 c^2 m_a^2 \left[b^2(s-b)^2 m_c^2 + c^2(s-c)^2 m_b^2\right].$$

In order to prove inequality (3.15), we next first compute E_0 in terms of R, r and s. With the help of MAPLE software, using formula (3.4)

and $s = (a+b+c)/2$, one can easily obtain the following identity:

$$\begin{aligned}
128E_0 =& 12b^7c^3a^6 + 2b^5c^5a^6 + 12b^6c^3a^7 + 2b^9c^2a^5 + 6b^7c^5a^4 \\
&- 2b^9c^3a^4 + 5b^6c^8a^2 - 12b^{10}c^3a^3 + 12b^7c^6a^3 + 6b^7c^4a^5 \\
&+ 6b^5c^4a^7 + 2b^8c^4a^4 - 12b^7c^7a^2 + 2b^9c^5a^2 + 2b^6c^5a^5 \\
&+ 12b^6c^7a^3 + 10b^4c^6a^6 + 10b^6c^6a^4 - 8b^5c^{11} + 2b^{10}c^6 \\
&+ 2b^5c^8a^3 + 2b^8c^5a^3 + 6b^5c^7a^4 - b^{10}c^4a^2 - b^4c^{10}a^2 \\
&+ 6b^4c^7a^5 + 2b^5c^9a^2 - 2b^4c^9a^3 - 2b^9c^4a^3 + 2b^4c^8a^4 \\
&- 2b^{12}c^2a^2 - 2b^2c^{12}a^2 + 10b^6c^4a^6 + 5b^6c^2a^8 - 12b^2c^7a^7 \\
&- 12b^7c^2a^7 - b^{10}c^2a^4 + 5b^8c^2a^6 + 5b^2c^6a^8 + 5b^2c^8a^6 \\
&- b^2c^{10}a^4 + 2b^2c^9a^5 + 4b^{13}c^2a + 4b^2c^{13}a - 4b^4c^{11}a \\
&- 4b^3c^{12}a + 8b^8c^7a + 8b^7c^8a - 4b^{12}c^3a + 4b^{10}c^5a \\
&- 8b^6c^9a - 4b^{11}c^4a + 4b^5c^{10}a - 8b^9c^6a - 2b^3c^9a^4 \\
&+ 12b^3c^7a^6 + 12b^3c^6a^7 + 2b^5c^3a^8 + 4b^7c^9 - 4b^8c^8 \\
&+ 2b^{12}c^4 + 2b^3c^5a^8 + 6b^4c^5a^7 + 2b^4c^4a^8 - 2b^{14}c^2 \\
&- 2b^2c^{14} - 2a^2c^{14} + 2a^4c^{12} + 4a^9c^7 + 2a^{10}c^6 - 4a^8c^8 \\
&- 12b^3c^{10}a^3 + 5b^8c^6a^2 + 2a^6c^{10} + 4a^{13}c^3 - 2a^{14}c^2 \\
&+ 4b^3c^{13} + 2b^4c^{12} + 4b^9c^7 + 4b^{13}c^3 - 8b^{11}c^5 + 4a^3c^{13} \\
&+ 4a^7c^9 - 8a^5c^{11} - 8a^{11}c^5 + 2a^{12}c^4 + 4a^7b^9 + 2a^5c^8b^3 \\
&- 2a^9c^4b^3 - 2a^9c^3b^4 + 2a^5c^3b^8 + 2b^6c^{10} - 2a^{12}c^2b^2 \\
&+ 4a^2c^{13}b + 4a^{13}c^2b - 4a^3c^{12}b - 8a^6c^9b + 8a^8c^7b \\
&+ 4a^{10}c^5b - 8a^9c^6b - 12a^{10}c^3b^3 - a^{10}c^4b^2 + 2a^9c^5b^2 \\
&+ 8a^7c^8b + 4a^5c^{10}b - 4a^4c^{11}b + 2a^{10}b^6 + 4a^9b^7 \\
&+ 2a^6b^{10} - 2a^2b^{14} - 4a^8b^8 + 4a^3b^{13} - 8a^{11}b^5 \\
&+ 2a^{12}b^4 + 4a^{13}b^3 - a^{10}c^2b^4 - 4a^{12}c^3b - 4a^{11}c^4b \\
&+ 2a^9c^2b^5 + 2a^4b^{12} - 2a^{14}b^2 - 8a^5b^{11} + 4a^2b^{13}c \\
&- 4a^3b^{12}c + 8a^8b^7c - 8a^9b^6c + 4a^{10}b^5c + 4a^5b^{10}c \\
&+ 8a^7b^8c - 8a^6b^9c - 4a^4b^{11}c + 4a^{13}b^2c - 4a^{11}b^4c \\
&- 4a^{12}b^3c + 2b^5c^6a^5. \qquad (3.16)
\end{aligned}$$

We continuously use symbols in Lemma 6 and 7. By calculation we obtain identity:

$$\begin{aligned}128E_0 =&2p_1d^5 + (6p_1p_3 - 4p_4 + 10q_2)d^4 + (2p_2p_5 - 24p_7\\&+ 12p_3p_4 - 2p_1p_6)d^3 + (5p_4p_6 + 2p_3p_7 - 8p_{10}\\&- 12q_5 - p_2p_8)d^2 + (4p_1p_{12} + 4p_4p_9 - 8p_5p_8\\&+ 8p_6p_7 - 4p_2p_{11} - 4p_3p_{10})d + 2p_4p_{12} + 4p_3p_{13}\\&+ 2p_6p_{10} - 4q_8 - 2p_{16} - 8p_5p_{11} + 4p_7p_9 - 2p_2p_{14}. \qquad (3.17)\end{aligned}$$

With the help of MAPLE software, using $d = abc = 4Rrs$ and related identities given in Lemma 6 and 7, we further obtain the following identity:

$$E_0 = 2s^2r^4F_0, \qquad (3.18)$$

where

$$\begin{aligned}F_0 =& - s^{10} + (8R^2 + 48Rr + 13r^2)s^8 - 2(124R^3 + 312R^2r\\&+ 76Rr^2 - 7r^3)rs^6 + 2(80R^5 + 1312R^4r + 1584R^3r^2\\&+ 574R^2r^3 + 44Rr^4 - 7r^5)rs^4 - (176R^4 + 472R^3r\\&+ 328R^2r^2 + 96Rr^3 + 13r^4)(4R + r)^2r^2s^2\\&+ (2R + r)^2(4R + r)^5r^3.\end{aligned}$$

Finally, by using identities (3.3) and (3.18), we can obtain the following identity:

$$D_0 = r^4s^2G_0, \qquad (3.19)$$

where

$$\begin{aligned}G_0 =&s^{12} + (96R^2 + 150Rr + 108r^2)s^{10} - (384R^4 + 4608R^3r\\&+ 7752R^2r^2 + 4854Rr^3 + 867r^4)s^8 + (512R^6 + 16704R^5r\\&+ 83136R^4r^2 + 122768R^3r^3 + 67476R^2r^4 + 9132Rr^5\\&- 1008r^6)s^6 - 3(5888R^7 + 73472R^6r + 211328R^5r^2\\&+ 250512R^4r^3 + 138808R^3r^4 + 33404R^2r^5 + 1780Rr^6\\&- 313r^7)rs^4 + 6(1792R^6 + 8560R^5r + 15496R^4r^2\end{aligned}$$

$$+13408R^3r^3+5938R^2r^4+1381Rr^5+150r^6)(4R+r)^2r^2s^2$$
$$-(80R^2+150Rr+73r^2)(2R+r)^2(4R+r)^5r^3.$$

Therefore, for proving inequality (3.15) we need to prove that

$$G_0 \geq 0. \tag{3.20}$$

We now recall that for any triangle ABC the following fundamental triangle inequality

$$t_0 \equiv -s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3 \geq 0 \tag{3.21}$$

holds (for the proofs, see for example, [3] and [11]). Moreover, from this inequality one can obtain the previous Gerretsen inequality (2.48) and another Gerretsen inequality (see [3]):

$$y_0 \equiv 4R^2+4Rr+3r^2-s^2 \geq 0. \tag{3.22}$$

Based on inequalities (3.21), (3.22), (2.48) and Euler's inequality (2.47), after analysis we obtain the following identity:

$$G_0=(y_0x_0^3+m_1)t_0+H_0x_0^3+m_2x_0^2+(R-2r)(m_3x_0+m_4), \tag{3.23}$$

where

$$\begin{aligned}
m_1 =& 72r^4(2862R^4+15795R^3r+5554R^2r^2+3328r^4),\\
m_2 =& 576Rr(12R^6-28R^5r+571R^4r^2+1566r^6),\\
m_3 =& 144r^3(2240R^6+9540R^5r-8844R^4r^2+18359R^3r^3\\
&-9790R^2r^4+6136Rr^5+4336r^6),\\
m_4 =& 576r^5(5700R^6+12915R^5r+5238R^4r^2-32452R^3r^3\\
&+9116R^2r^4-3088Rr^5-2704r^6),\\
H_0 =& (104R^2+222Rr+94r^2)s^4-(400R^4+160R^3r+588R^2r^2\\
&+2944Rr^3+2332r^4)s^2+512R^6-1472R^5r+15616R^4r^2\\
&-576R^3r^3-51152R^2r^4-28814Rr^5+26950r^6.
\end{aligned}$$

Note that

$$12R^6-28R^5r+571R^4r^2=R^4(12R^2-28Rr+571r^2)>0,$$

so that $m_2>0$. And, by Euler's inequality $R \geq 2r$ we have

$$9540R^5r-8844R^4r^2=12R^4r(795R-737r)>0$$

and

$$18359R^3r^3 - 9790R^2r^4 = 11R^2r^3(1669R - 890r) > 0,$$

so that $m_3 > 0$. In addition, it is easy to check that

$$\begin{aligned}
&5700R^6 + 12915R^5r + 5238R^4r^2 - 32452R^3r^3 + 9116R^2r^4 \\
&- 3088Rr^5 - 2704r^6 \\
&= 5700e^6 + 81315e^5r + 476388e^4r^2 + 1438052e^3r^3 \\
&+ 2341316e^2r^4 + 1939168er^5 + 629856r^6.
\end{aligned} \tag{3.24}$$

Thus $m_4 > 0$ follows from Euler's inequality $e \geq 0$. Therefore, from identity (3.23) by the fundamental triangle inequality $t_0 \geq 0$, Gerretsen's inequalities $x_0 \geq 0$ and $y_0 \geq 0$, it remains to prove the strict inequality $H_0 > 0$, i.e.,

$$H_0 \equiv a_0s^4 + b_0s^2 + c_0 > 0, \tag{3.25}$$

where

$$\begin{aligned}
a_0 =&104R^2 + 222Rr + 94r^2, \\
b_0 =&- (400R^4 + 160R^3r + 588R^2r^2 + 2944Rr^3 + 2332r^4), \\
c_0 =&512R^6 - 1472R^5r + 15616R^4r^2 - 576R^3r^3 - 51152R^2r^4 \\
&- 28814Rr^5 + 26950r^6.
\end{aligned}$$

We set

$$\begin{aligned}
I_0 =&(104R^2 + 222Rr + 94r^2)s^2 - 400R^4 + 1504R^3r + 2444R^2r^2 \\
&- 2550Rr^3 - 2802r^4.
\end{aligned}$$

Next, we consider two cases to complete the proof of inequality (3.25).

Case 1. $I_0 > 0$.

It is easy to check that

$$H_0 = x_0I_0 + J_0, \tag{3.26}$$

where

$$\begin{aligned}
J_0 =&512R^6 - 7872R^5r + 41680R^4r^2 + 31008R^3r^3 \\
&- 104172R^2r^4 - 60896Rr^5 + 40960r^6.
\end{aligned}$$

Thus, by Gerretsen's inequality $x_0 \geq 0$, it remains to show prove $J_0 > 0$ under Case 1. Note that J_0 can be rewritten as

$$\begin{aligned} J_0 =&16e^3(32e^3 - 108e^2r - 395er^2 + 8218r^3) + 575316e^2r^4 \\ &+ 696816er^5 + 198288r^6, \end{aligned} \tag{3.27}$$

where $e = R - 2r \geq 0$. So, we only need to show that

$$32e^3 - 108e^2r - 395er^2 + 8218r^3 > 0.$$

To do this, we can assume that $r = 1$ and we have to show

$$32e^3 - 108e^2 - 395e + 8218 > 0, \tag{3.28}$$

which can be rewritten as

$$32(e^3 - 4e^2 - 15e + 64) + 20e^2 + 85e + 6170 > 0.$$

It remains to show

$$e^3 - 4e^2 - 15e + 64 > 0. \tag{3.29}$$

Putting $e = 3 + t$, then it becomes

$$t^3 + 5t^2 - 12t + 10 > 0.$$

Note that $5t^2 - 12t + 10 > 0$, thus the above inequality holds for $t > 0$ and then inequality (3.29) is proved when $e > 3$. On the other hand, inequality (3.29) can be rewritten as

$$(3 - e)(-e^2 + e + 18) + 10 > 0,$$

which is clear true for $0 < e < 3$. Therefore, we deduce that inequality (3.29) holds for any positive real number $e > 0$. And, inequality (3.25) is proved under Case 1.

Case 2. $I_0 \leq 0$.

Note that H_0 is a quadratic function (in s^2) with positive quadratic term. According to the property of quadratic functions with one variable, to prove $H_0 > 0$ under Case 2 we need to show $c_0 > 0$ and $F_i < 0$, where F_i is the discriminant of H_0 given by $F_i = b_0^2 - 4a_0c_0$.

By the assumption $I_0 \leq 0$ and Gerretsen's inequality (2.48) we have

$$\begin{aligned} &(104R^2 + 222Rr + 94r^2)(16Rr - 5r^2) - 400R^4 + 1504R^3r + 2444R^2r^2 \\ &- 2550Rr^3 - 2802r^4 \leq 0. \end{aligned}$$

Simplifying gives us

$$K_0 \equiv -400R^4 + 3168R^3r + 5476R^2r^2 - 2156Rr^3 - 3272r^4 \leq 0. \quad (3.30)$$

Through analysis, we obtain the following identity:

$$\begin{aligned} 15625c_0 = &- K_0(20000R^2 + 100900Rr + 1682928r^2) \\ &+ 2r^3(2915962152e^3 + 21562514576e^2r \\ &+ 49054134321er^2 + 32643184221r^3), \end{aligned} \quad (3.31)$$

where $e = R - 2r \geq 0$. Since $K_0 \leq 0$, we have $c_0 > 0$.

On the other hand, it is easy to compute the discriminant F_i as follows:

$$\begin{aligned} F_i = &- 52992R^8 + 285696R^7r - 4885632R^6r^2 - 10530560R^5r^3 \\ &+ 19072528R^4r^4 + 61834560R^3r^5 + 45018352R^2r^6 \\ &+ 633280Rr^7 - 4694976r^8. \end{aligned} \quad (3.32)$$

Also, it is easy to check the following identity:

$$9765625F_i = K_0L_0 - 64r^5M_0, \quad (3.33)$$

where

$$\begin{aligned} L_0 =& 1293750000R^4 + 3271500000R^3r + 162899842500R^2r^2 \\ &+ 1585074025100Rr^3 + 14290030971992r^4, \\ M_0 =& 827889280426379e^3 + 6121431766559877e^2r \\ &+ 13988526742361367er^2 + 9384579224935542r^3. \end{aligned}$$

Since $L_0 > 0$, $M_0 > 0$ and $K_0 \leq 0$, we deduce that $F_i < 0$ from identity (3.33) and inequality $I_0 > 0$ is proved under Case 2.

Combining the arguments of the above two cases, we proved that the strict inequality (3.25) holds for all triangles. Therefore, inequalities (3.20), (3.15), (3.12) and (1.11) are proved. It is easy to determine that the equality in (3.14) holds if and only if ABC is equilateral and we then further easily deduce that the equality in (1.11) holds if and only if triangle ABC is equilateral and P is its center. This completes the proof of Theorem 2.

4. **Proof of Theorem 3**

In this section, we shall prove Theorem 3. The following two lemmas will be used in the proof.

Lemma 10. *For an interior point P of triangle ABC the following inequality holds:*

$$aR_1 \geq br_3 + cr_2, \tag{4.1}$$

with equality if and only if P lies on the line AO (O is the circumcenter of triangle ABC).

We can find various proofs of inequality (4.1) from a lot of papers (see, for example [12]-[14]) related to the famous Erdös-Mordell inequality:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3). \tag{4.2}$$

Lemma 11. *In any triangle ABC the following inequality holds:*

$$(m_b + m_c)^2 \geq \frac{9}{4}a^2 + h_a^2, \tag{4.3}$$

where h_a is the altitude from vertex A to the side BC. Equality in (4.3) *holds if and only if $b = c$.*

For a proof of inequality (4.3), see [15].

Next, we prove Theorem 3.

Proof

According to Lemma 10, to prove inequality (1.12) we need to show

$$\sum m_a^2 \frac{br_3 + cr_2}{ar_1} \geq 2 \sum m_a^2,$$

which is equivalent to

$$\sum a \left(\frac{r_3}{br_2} m_b^2 + \frac{r_2}{cr_3} m_c^2 \right) \geq 2 \sum m_a^2.$$

Since

$$\frac{r_3}{br_2} m_b^2 + \frac{r_2}{cr_3} m_c^2 \geq \frac{2m_b m_c}{\sqrt{bc}},$$

We only need to prove that

$$\sum \frac{a}{\sqrt{bc}} m_b m_c \geq \sum m_a^2. \tag{4.4}$$

Note that $b+c \geq 2\sqrt{bc}$. We shall show the following stronger inequality:

$$\sum \frac{a}{b+c} m_b m_c \geq \frac{1}{2} \sum m_a^2. \tag{4.5}$$

It follows from Lemma 11 that

$$2m_b m_c \geq \frac{9}{4}a^2 + h_a^2 - m_b^2 - m_c^2.$$

Using the previous median formula (3.4) we further obtain

$$m_b m_c \geq \frac{1}{8}(5a^2 - b^2 - c^2 + 4h_a^2). \tag{4.6}$$

Thus, to prove inequality (4.5) we only need to prove

$$\sum \frac{a}{b+c}(5a^2 - b^2 - c^2 + 4h_a^2) \geq 3\sum a^2. \tag{4.7}$$

Using the known formula:

$$h_a = \frac{\sqrt{2\sum b^2c^2 - \sum a^4}}{2a}, \tag{4.8}$$

we easily get

$$\begin{aligned} &\sum \frac{a}{b+c}(5a^2 - b^2 - c^2 + 4h_a^2) - 3\sum a^2 \\ &= \frac{N_0}{abc(b+c)(c+a)(a+b)}, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} N_0 =& 2\sum b^4c^4 + 2abc\sum a^5 - 2(abc)^2\sum bc - \sum a^2(b^6 + c^6) \\ &+ abc\sum a^2(b^3 + c^3) - abc\sum a(b^4 + c^4). \end{aligned}$$

It remains to prove that

$$N_0 \geq 0. \tag{4.10}$$

Putting $b+c-a = 2x, c+a-b = 2y, a+b-c = 2z$, the we have $a = y+z, b = z+x$ and $c = x+y$. It is easy to obtain the following

identity:

$$\begin{aligned}4N_0 = &\sum x(y^7+z^7)+\sum x^2(y^6+z^6)-\sum x^3(y^5+z^5)\\&+2xyz\sum x(y^4+z^4)-6xyz\sum x^2(y^3+z^3)\\&-2\sum y^4z^4+2(xyz)^2\sum yz.\end{aligned}\tag{4.11}$$

After analysis, we obtain the following identity:

$$\begin{aligned}8N_0 =&4xyz\sum x\sum x^2(x-y)(x-z)\\&+2\sum yz(x^2+4zx+4xy+y^2+z^2+yz)(y-z)^2(y+z-x)^2\\&+xyz\sum[3x(y^2+z^2+4yz)+(y+z)(3x^2+5yz)](y-z)^2,\end{aligned}\tag{4.12}$$

Note that Schur's inequality (cf. [5, Theorem 80]):

$$\sum x^2(x-y)(x-z)\geq 0,\tag{4.13}$$

we deduce $N_0 \geq 0$ from (4.12). Therefore, inequalities (4.5) and (1.12) are proved. Moreover, it is easy to determine that equality in (1.12) holds if and only triangle ABC is equilateral and P is its center. This completes the proof of Theorem 3.

5. Three conjectures

In the last section, we propose three conjectures as open problems.

It is natural to consider unified exponential generalizations of inequality (1.6) and (1.12). After checking by the computer, we propose the following conjecture:

Conjecture 1. *Let k be a real number such that $-1 \leq k \leq 3$. Then for an interior point P of triangle ABC the following inequality holds:*

$$m_a^k\frac{R_1}{r_1}+m_b^k\frac{R_2}{r_2}+m_c^k\frac{R_3}{r_c}\geq 2(m_a^k+m_b^k+m_c^k).\tag{5.1}$$

For inequality (4.4), we propose the following exponential generalization:

Conjecture 2. *Let k be a real number such that $0 < k \leq \frac{3}{2}$. Then for any triangle ABC the following inequality holds:*

$$\frac{a}{\sqrt{bc}}(m_b m_c)^k + \frac{b}{\sqrt{ca}}(m_c m_a)^k + \frac{c}{\sqrt{ab}}(m_a m_b)^k \\ \geq m_a^{2k} + m_b^{2k} + m_c^{2k}. \tag{5.2}$$

If this conjecture is true, then from the proof of Theorem 1 we can easily deduce that inequality (5.1) holds when $0 < k \leq 3$.

For the acute triangle ABC, the author conjectures that inequality (1.11) can be improved to the following:

Conjecture 3. *For an interior point P of acute triangle ABC the following inequality holds:*

$$m_a \frac{R_1}{r_1} + m_b \frac{R_2}{r_2} + m_c \frac{R_3}{r_3} \geq 9R. \tag{5.3}$$

Remark 3. *The author has known that inequality* (5.3) *can not be proved by using the method to prove inequality* (1.11).

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RINGS IN WHICH ELEMENTS ARE A SUM OF A CENTRAL AND NILPOTENT ELEMENT

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Abstract

In this paper, we introduce a new class of rings whose elements are a sum of a central element and a nilpotent element, namely, a ring R is called CN if each element a of R has a decomposition $a = c + n$ where c is central and n is nilpotent. In this note, we characterize elements in $M_n(R)$ and $U_2(R)$ having CN-decompositions. For any field F, we give examples to show that $M_n(F)$ can not be a CN-ring. For a division ring D, we prove that if $M_n(D)$ is a CN-ring, then the cardinality of the center of D is strictly greater than n. Especially, we investigate several kinds of conditions under which some subrings of full matrix rings over CN rings are CN.

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1 Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. $\mathrm{Inv}(R)$, $J(R)$, $C(R)$ and $\mathrm{nil}(R)$ will denote the group of units, the Jacobson radical, the center and the set of all nilpotent elements of a ring R, respectively. Recall that in [2], uniquely nil clean rings are defined. An element a in a ring R is called *uniquely nil clean* if there is a unique idempotent $e \in R$ such that $a - e$ is nilpotent. The ring R is *uniquely nil clean* if each of its elements is uniquely clean. It is proved that in a uniquely nil clean ring, every idempotent is central. Also a uniquely nil clean ring R is called *uniquely strongly nil clean* [5] if a and e commute. Strongly nil cleanness and uniquely strongly nil cleanness are equivalent by [2]. Let R be a (*)-ring. In [7], $a \in R$ is called *uniquely strongly nil *-clean ring* if there is a unique projection $p \in R$, i.e., $p^2 = p = p^*$, and $n \in \mathrm{nil}(R)$ such that $a = p + n$ and $pn = np$. R is called a *uniquely strongly nil *-clean ring* if each of its elements is uniquely strongly nil *-clean. Another version of the notion of clean rings is that of CU rings. In [1], an element $a \in R$ is called a *CU element* if there exist $c \in C(R)$ and $n \in \mathrm{nil}(R)$ such that $a = c + n$. The ring R is called *CU* if each of its elements is CU. Motivated by these facts, we investigate basic properties of rings in which every element is the sum of a central element and a nilpotent element.

In what follows, $\mathbb{Z}_n$ is the ring of integers modulo n for some positive integer n. Let $M_n(R)$ denote the full matrix ring over R and $U_n(R)$ stand for the subring of $M_n(R)$ consisting of all $n \times n$ upper triangular matrices. And in the following, we give definitions of some other subrings of $U_n(R)$ to discuss in the sequel whether they satisfy CN property:

$$D_n(R) = \{(a_{ij}) \in M_n(R) \mid \text{ all diagonal entries of } (a_{ij}) \text{ are equal}\},$$

$$V_n(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{n} a_j e_{(i-j+1)i} \mid a_j \in R \right\},$$

$$V_n^k(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{k} x_j e_{(i-j+1)i} + \sum_{i=j}^{n-k} \sum_{j=1}^{n-k} a_{ij} e_{j(k+i)} : x_j, a_{ij} \in R \right\}$$

where $x_i \in R$, $a_{js} \in R$, $1 \leq i \leq k$, $1 \leq j \leq n-k$ and $k+1 \leq s \leq n$,

$$D_n^k(R) = \{\left\{\sum_{i=1}^{k}\sum_{j=k+1}^{n} a_{ij}e_{ij} + \sum_{j=k+2}^{n} b_{(k+1)j}e_{(k+1)j} + cI_n \mid a_{ij}, b_{ij}, c \in R\right\}$$

where $k = [n/2]$, i.e., k satisfies $n = 2k$ when n is an even integer, and $n = 2k+1$ when n is an odd integer, and

$$D_n^{\mathbb{Z}}(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = a_{nn} \in \mathbb{Z}, a_{ij} \in R, \{i,j\} \subseteq \{2,3,\ldots,n-1\}\}.$$

2 Basic Properties and Examples

Definition 2.1. Let R be a ring with identity. An element $a \in R$ is called *CN* or *it has a CN-decomposition* if $a = c + n$, where $c \in C(R)$ and $n \in \mathrm{nil}(R)$. If every element of R has a CN decomposition, then R is called *a CN ring.*

We present some examples to illustrate the concept of CN property for rings.

Example 2.2. (1) Every commutative ring is CN.

(2) Every nilpotent element in a ring R has a CN decomposition.

(3) For a field F and for any positive integer n, $D_n(F)$ is a CN ring.

Proposition 2.3. *Let R be a ring and n a positive integer. Then $A \in M_n(R)$ has a CN decomposition if and only if for each $P \in GL_n(R)$, $PAP^{-1} \in M_n(R)$ has a CN decomposition.*

Proof. Assume that $A \in M_n(R)$ has a CN decomposition $A = C + N$ where $C \in C(M_n(R))$ and $N \in \mathrm{nil}(M_n(R))$. Then $PAP^{-1} = PCP^{-1} + PNP^{-1}$ is a CN decomposition of PAP^{-1} since $PCP^{-1} = C \in C(M_n(R))$ and it is obvious that $PNP^{-1} \in \mathrm{nil}(M_n(R))$. Conversely, suppose that PAP^{-1} has a CN decomposition $PAP^{-1} = C + N$. Then $A = P^{-1}CP + P^{-1}NP$ is the CN decomposition of PAP^{-1}. □

Let R be a commutative ring and n a positive integer. The following result gives us a way to find out whether $A \in M_n(R)$ has a CN decomposition. Note that it is easily shown that for a commutative ring $A \in C(M_n(R))$ if and only if $A = cI_n$ for some $c \in R$.

Theorem 2.4. *Let R be a commutative ring. Then $A \in M_n(R)$ has a CN decomposition if and only if $A - cI_n \in nil(M_n(R))$ for some $c \in R$.*

Proof. Assume that $A \in M_n(R)$ has a CN decomposition. By assumption there exists $c \in R$ such that $A - cI_n \in \mathrm{nil}(M_n(R))$. Conversely, suppose that for any $A \in M_n(R)$, there exists $c \in R$ such that $A - cI_n \in \mathrm{nil}(M_n(R))$. Since cI_n is central in $M_n(R)$, $A \in M_n(R)$ has a CN decomposition. □

Remark. Let R be a commutative ring. Then $A \in M_n(R)$ is a nilpotent matrix if and only if all eigenvalues of A are zero. A ring R is *reduced* if R has no nonzero nilpotent element. Hence we have.

Corollary 2.5. *Let R be a commutative reduced ring and n a positive integer. Then $A \in M_n(R)$ has a CN decomposition if and only if the only eigenvalue for $A - cI_n$ is 0 for some $c \in R$.*

Proposition 2.6. *Let R be a commutative ring. Then $U_2(R)$ is a CN ring if and only if for any $a, b \in R$, there exists $c \in R$ such that $a - c$, $b - c \in nil(R)$.*

Proof. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(R)$ has CN decomposition if and only if there exist $C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in C(M_2(R))$ and $N = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in \mathrm{nil}(M_2(\mathbb{R}))$ such that $A = C + N$. Since $N \in \mathrm{nil}(M_2(R))$ if and only if $x, z \in \mathrm{nil}(R)$, $A = C + N$ is the CN decomposition of A if and only if there exists $c \in R$ such that $A - cI \in \mathrm{nil}(M_2(R))$ if and only if $a - c$, $b - c \in \mathrm{nil}(R)$. □

Example 2.7. Let $R = \mathbb{Z}$ and $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \in U_2(R)$. Then there is no $c \in \mathbb{Z}$ such that $3 - c$ and $5 - c$ are nilpotent. By Proposition 2.6, $U_2(\mathbb{Z})$ is not CN.

Theorem 2.8. *Let R be a commutative local ring. If $M_2(R)$ is a CN ring, then $R/J(R)$ is not isomorphic to $\mathbb{Z}_2$.*

Proof. Assume that $M_2(R)$ is a CN ring. Suppose that $R/J(R)$ is isomorphic to $\mathbb{Z}_2$ and we get a contradiction. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(R)$ and $f(c) = det(A - cI_2)$ be the characteristic polynomial of A. Then $f(c) = c(c-1) \in \text{nil}(R)$. By Proposition 2.6, $1-c$ and c are nilpotent. Since $1 = c + (1-c)$, By hypothesis, c or $1-c$ is invertible, therefore $c \in J(R)$ or $1 - c \in J(R)$. This is a contradiction. □

In [1], Chen and at al. defined and studied CU rings. Let R be a ring. An element $a \in R$ has a CU-decomposition if $a = c + u$ for some $c \in C(R)$ and $u \in U(R)$. A ring R is called CU, if every element of R has a CU-decomposition.

Proposition 2.9. *Every CN ring is CU.*

Proof. Let R be a CN ring and $a \in R$. By assumption $a + 1 = c + n$ for some $c \in C(R)$ and $n \in N(R)$. Hence $a = c + (n-1)$ is a CU decomposition of a. □

Theorem 2.10. *Let R be a division ring and n a positive integer. If $M_n(R)$ is a CN ring, then $|C(R)| > n$.*

Proof. Assume that $|C(R)| < n$. Consider A as a diagonal matrix which has the property that each element of $C(R)$ is one of the diagonal entries of A. For such a matrix A there is no $c \in C(R)$ for which $A - cI$ is a unit. Hence $M_n(R)$ is not a CU ring. By Proposition 2.9, $M_n(R)$ can not be a CN ring. This contradicts hypothesis. So $|C(R)| > n$. □

The converse of Proposition 2.9 is not true in general.

Example 2.11. Let $\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$ be the ring of real quaternions, where $i^2 = j^2 = k^2 = ijk = 1$ and $ij = -ji, ik = -ki, jk = -kj$. $\mathbb{H}$ is a noncommutative division ring. Note that $C(\mathbb{H}) = \mathbb{R}$ and $nil(\mathbb{H}) = 0$. Let $a \in \mathbb{H}$. If $a = 0$, then $0 = 1 + (-1)$ is the CU-decomposition. If $a \neq 0$, then $a = 0 + a$ is the CU-decomposition of a. Hence $\mathbb{H}$ is a CU ring. On the other hand there is no CN decomposition of $i \in \mathbb{H}$. Hence it is not a CN ring.

Example 2.12. Let D be a division ring and consider the ring $D_2(D)$. The ring $D_2(D)$ is a noncommutative local ring, and so it is a CU-ring, but not a CN ring.

For a positive integer n, one may suspect that if R is a CN ring then the matrix ring $M_n(R)$ is also CN. The following examples shows that this is not true in general. Also whether or not $M_n(R)$ to be a CN ring does not depend on the cardinality of $C(R)$ comparing with n, that is, $|C(R)| \geq n$ or $|C(R)| < n$.

Example 2.13. (1) Since $\mathbb{Z}$ is commutative, it is a CN ring. But $R = M_2(\mathbb{Z})$ is not a CN ring.
(2) $R = M_2(\mathbb{Z}_3)$ is not a CN ring.
(3) $R = M_3(\mathbb{Z}_2)$ is not a CN ring.

Proof. (1) Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \in M_2(\mathbb{Z})$ which is neither central nor nilpotent. Let $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \in C(M_2(\mathbb{Z}))$ and $N = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in \mathrm{nil}(M_2(\mathbb{Z}))$ with $A = C + N$. Then $x + t = 0$ and $zy = xt$. This is a contradiction. Hence A does not have CN decomposition.
(2) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Z}_3)$ which is neither central nor nilpotent. Assume that A has CN decomposition with $A = C + N$ where $C = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in C(M_2(\mathbb{Z}_3))$ and $N = \begin{bmatrix} x & y \\ t & u \end{bmatrix} \in \mathrm{nil}(M_2(\mathbb{Z}_3))$. $A = C + N$ implies $1 = a + x$, $0 = a + u$ and $y = t = 0$. These equalities do not satisfied in $\mathbb{Z}_3$. For if $a = 0$, then $x = 1$; if $a = 1$, then $x = 0$ and $u = 2$; if $a = 2$, then $x = 2$ and $u = 1$. All these lead us a contradition. Hence $M_2(\mathbb{Z}_3)$ is not a CN ring.
(3) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_3(\mathbb{Z}_2)$ which is neither central nor nilpotent. Assume that A has CN decomposition with $A = C + N$ where $C = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in C(M_3(\mathbb{Z}_2))$ and $N = \begin{bmatrix} x & y & z \\ t & u & v \\ k & l & m \end{bmatrix} \in \mathrm{nil}(M_3(\mathbb{Z}_2))$. $A = C + N$

implies $1 = a + x$, $0 = a + u$, $0 = a + m$ and $y = z = v = t = k = l = 0$. These equalities do not satisfied in $\mathbb{Z}_2$. Hence $M_3(\mathbb{Z}_2)$ is not a CN ring. In fact, assume that $1 = a + x$ holds in $\mathbb{Z}_2$. There are two cases for a. $a = 0$ or $a = 1$. If $a = 1$ then $x = 0$ and $u = 1$. N being nilpotent implies $u = 1$ is nilpotent. A contradiction. Otherwise, $a = 0$. Then $x = 1$. Again N being nilpotent implies $x = 1$ is nilpotent. A contradiction. Thus $M_3(\mathbb{Z}_2)$ is not a CN ring. □

In spite of the fact that $U_n(R)$ need not be *CN* for any positive integer n, there are *CN* subrings of $U_n(R)$.

Proposition 2.14. *For a ring R and an integer $n \geq 1$, the following are equivalent:*
(1) *R is CN.*
(2) *$D_n(R)$ is CN.*
(3) *$D_n^k(R)$ is CN.*
(4) *$V_n(R)$ is CN.*
(5) *$V_n^k(R)$ is CN.*

Proof. Note that the elements of $D_n(R)$, $D_n^k(R)$, $V_n(R)$ and $V_n^k(R)$ having zero as diagonal entries are nilpotent. To complete the proof, it is enough to show (1) holds if and only if (2) holds for $n = 4$. The other cases are just a repetition.

(1) $\Rightarrow$ (2) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_5 & a_6 \\ 0 & 0 & a_1 & a_7 \\ 0 & 0 & 0 & a_1 \end{bmatrix} \in D_4(R)$. By (1), there exist $c \in C(R)$ and $n \in \mathrm{nil}(R)$ such that $a_1 = c + n$.

Let $C = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$ and $N = \begin{bmatrix} n & a_2 & a_3 & a_4 \\ 0 & n & a_5 & a_6 \\ 0 & 0 & n & a_7 \\ 0 & 0 & 0 & n \end{bmatrix}$. Then $C \in C(V_n(R))$ and $N \in \mathrm{nil}(D_n(R))$.

(2) $\Rightarrow$ (1) Let $a \in R$. By (2) $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \in D_4(R)$ has a CN

decomposition $A = C + N$ where $C = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \in C(D_4(R))$ and $N = \begin{bmatrix} n & * & * & * \\ 0 & n & * & * \\ 0 & 0 & n & * \\ 0 & 0 & 0 & n \end{bmatrix} \in C(D_n(R))$. Then $a = c + n$ with $c \in C(R)$ and $n \in$ nil(R). □

Lemma 2.15. *Every homomorphic image of CN ring is CN ring.*

Proof. Let $f : R \to S$ be an epimorphism of rings with R CN ring. Let $s = f(x) \in S$ with $x \in R$. There exist $c \in C(R)$ and $n \in$ nil(R) such that $x = c + n$. Since f is epic, $f(c) \in C(S)$ and $f(n) \in$ nil(R). Hence $s = f(c) + f(n)$ is CN decomposition of s. □

Proposition 2.16. *Let $R = \prod_{i \in I} R_i$ be a direct product of rings. R is CN if and only if R_i is CN for each $i \in I$.*

Proof. We may assume that $I = \{1, 2\}$ and $R = R_1 \times R_2$. Note that $C(R) = C(R_1) \times C(R_2)$ and $nil(R) = nil(R_1) \times nil(R_2)$.
Necessity: Let $r_1 \in R_1$. Then $(r_1, 0) = (c_1, c_2) + (n_1, n_2)$ where $(c_1, c_2) \in C(R)$ and $(n_1, n_2) \in nil(R)$. Hence $r_1 = c_1 + n_1$ is the CN decomposition of $r_1 \in R_1$. So R_1 is CN. A similar proof takes care for R_2 be CN.
Sufficiency: Assume that R_1 and R_2 are CN. Let $(r_1, r_2) \in R$. By assumption r_1 and r_2 have CN decompositions $r_1 = c_1 + n_1$ and $r_2 = c_2 + n_2$ where c_1 is central in R_1, n_1 is nilpotent in R_1 and c_2 is central in R_2, n_2 is nilpotent in R_2. Hence (r_1, r_2) has a CN decomposition $(r_1, r_2) = (c_1, c_2) + (n_1, n_2)$. This completes the proof. □

Let R be a ring and $D(\mathbb{Z}, R)$ denote the *Dorroh extension* of R by the ring of integers $\mathbb{Z}$ (see [3]). Then $D(\mathbb{Z}, R)$ is the ring defined by the direct sum $\mathbb{Z} \oplus R$ with componentwise addition and multiplication $(n, r)(m, s) = (nm, ns + mr + rs)$ where (n, r), $(m, s) \in D(\mathbb{Z}, R)$. It is clear that $C(D(\mathbb{Z}, R)) = \mathbb{Z} \oplus C(R)$. The identity of $D(\mathbb{Z}, R)$ is $(1, 0)$ and the set of all nilpotent elements is nil$(D(\mathbb{Z}, R)) = \{(0, r) \mid r \in \text{nil}(R)\}$.

Theorem 2.17. *Let R be a ring. Then R is a CN ring if and only if $D(\mathbb{Z}, R)$ is CN.*

Proof. Assume that R is CN. Let $(a,r) \in D(\mathbb{Z}, R)$. Since R is a CN ring, $r = c+n$ for some $c \in C(R)$ and $n \in \mathrm{nil}(R)$. Then $(a,r) = (a,c) + (0,n)$ is the CN decomposition of (a,r). Conversely, let $r \in R$. Then $(0,r) = (a,c)+(0,s)$ as a CN decomposition where $(n,c) \in C(D(\mathbb{Z}, R))$ and $(0,n) \in \mathrm{nil}(D(\mathbb{Z}, R))$. Then $c \in C(R)$ and $s \in \mathrm{nil}(R)$. It follows that $r = c+s$ is the CN decomposi tionof r. Hence R is CN. $\square$

Let R be a ring and S a subring of R and

$$T[R,S] = \{(r_1, r_2, \cdots, r_n, s, s, \cdots) : r_i \in R, s \in S, n \geq 1, 1 \leq i \leq n\}.$$

Then $T[R,S]$ is a ring under the componentwise addition and multiplication. Note that $\mathrm{nil}(T[R,S]) = T[\mathrm{nil}(R), \mathrm{nil}(S)]$ and $C([T,S]) = T[C(R), C(R) \cap C(S)]$.

Proposition 2.18. *R be a ring and S a subring of R. Then the following are equivalent.*

1. *$T[R,S]$ is CN.*
2. *R and S are CN.*

Proof. (1) $\Rightarrow$ (2) Assume that $T[R,S]$ is a CN ring. Let $a \in R$ and $X = (a,0,0,\dots) \in T[R,S]$. There exist a central element $C = (r_1, r_2, \cdots, r_n, s, s, \cdots)$ and a nilpotent element $N = (s_1, s_2, \cdots, s_k, t, t, \cdots)$ in $T[R,S]$ such that $X = C + N$. Then r_1 is in the center of R and s_1 is nilpotent in R and $a = r_1 + s_1$ is the CN decomposition of a. Hence R is CN. Let $s \in S$. By considering $Y = (0, s, s, s, \cdots) \in T[R,S]$, it can be seen that s has a CN decomposition.
(2) $\Rightarrow$ (1) Let R and S be CN rings and $Y = (a_1, a_2, \cdots, a_m, s, s, s, \cdots)$ be an arbitrary element in $T[R,S]$. Then there exist $c_i \in C(R)$, $1 \leq i \leq m$, $c \in C(R) \cap C(S)$ and $n_i \in \mathrm{nil}(R)$, $1 \leq i \leq m$, $t \in \mathrm{nil}(S)$ and such that $a_i = c_i + n_i$ for all $1 \leq i \leq m$ and $s = c+t$. Let $C = (c_1, c_2, \cdots, c_m, c, c, \cdots)$ and $N = (n_1, n_2, \cdots, n_m, t, t, \cdots)$. It is obvious that $C \in C(T[R,S])$ and $N \in \mathrm{nil}(T[R,S])$. Hence $Y = C + N$ is a CN decomposition of Y. $\square$

3 Some CN subrings of matrix rings

In this section, we study some subrings of full matrix rings whether or not they are CN rings. We first determine nilpotent and central elements of so-called subrings of matrix rings.

The rings $L_{(s,t)}(R)$: Let R be a ring, and $s, t \in C(R)$. Let $L_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R \right\}$, where the operations are defined as those in $M_3(R)$. Then $L_{(s,t)}(R)$ is a subring of $M_3(R)$.

Lemma 3.1. *Let R be a ring, and let s, t be in the center of R. Then the following hold.*

(1) The set of all nilpotent elements of $L_{(s,t)}(R)$ is

$$nil(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R) \mid a, d, f \in nil(R), c, e \in R \right\}.$$

(2) The set of all central elements of $L_{(s,t)}(R)$ is

$$C(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid sa = sd, td = tf, a, d, f \in C(R) \right\}.$$

Proof. (1) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in \mathrm{nil}(L_{(s,t)}(R)$. Assume that $A^n = 0$. Then $a^n = d^n = f^n = 0$. Conversely, Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in (L_{(s,t)}(R)$ with $a^{n_1} = 0$, $d^{n_1} = 0$ and $f^{n_1} = 0$ and $n = \max\{n_1, n_2, n_3\}$. Then $A^{n+1} = 0$.

(2) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in C(L_{(s,t)}(R)))$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ s & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \in L_{(s,t)}(R))$.

By $AB = BA$ implies $sc + sd = sa$ and $td = tf$ ....(*)

Let $C = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & t \\ 0 & 0 & 1 \end{bmatrix} \in L_{(s,t)}(R))$.

$AC = CA$ implies $sa = sd$ and $dt + te = tf$ ....(**).

(*) and (**) implies $sa = sd$ and $tf = td$. For the converse inclusion,

let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$ with $sa = sd$, $td = tf$ and a, d, $f \in C(R)$.

Let $B = \begin{bmatrix} x & 0 & 0 \\ sy & z & tu \\ 0 & 0 & v \end{bmatrix} \in L_{(s,t)}(R)$. Then $AB = \begin{bmatrix} ax & 0 & 0 \\ sdy & dz & e \\ 0 & 0 & tdu \end{bmatrix}$, $BA =$ $\begin{bmatrix} xa & 0 & 0 \\ sya & zd & tuf \\ 0 & 0 & vf \end{bmatrix}$. By the conditions; $sa = sd$, $td = tf$, $sc = 0$, $te = 0$ and a, d, $f \in C(R)$, $AB = BA$ for all $B \in L_{(s,t)}(R)$. Hence $A \in C(L_{(s,t)}(R))$. □

Consider following subrings of $L_{(s,t)}(R)$.

$$V_2(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{bmatrix} \in L_{(s,t)}(R) \mid a, e \in R \right\}$$

$$C(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R) \mid a, d, f \in C(R), c, e \in R, sa = sd, td = tf \right\}$$

It is easy to check that $V_2(L_{(s,t)}(R))$ and $C(L_{(s,t)}(R))$ are subrings of $L_{(s,t)}(R)$.

Proposition 3.2. *Let R be a ring. Following hold:*

(1) R is a CN ring if and only if $V_2(L_{(s,t)}(R))$ is a CN ring.

(2) $C(L_{(s,t)}(R))$ is a ring consisting of elements having CN decompositions.

(3) *Assume that R is a CN ring. If for any $\{a,d,f\} \subseteq R$ having a CN decomposition $a = x+p$, $d = y+q$ and $f = z+r$ with $\{x,y,z\} \subseteq C(R)$ and $\{p,q,r\} \subseteq nil(R)$ satisfy $sx = sy$ and $ty = tz$, then $L_{(s,t)}(R)$ is a CN ring.*

Proof. (1) Assume that R is a CN ring. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{bmatrix} \in V_2(L_{(s,t)}(R))$. There exist $c \in C(R)$ and $n \in \text{nil}(R)$ such that $a = c + n$. Then $C = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \in C(L_{(s,t)}(R))$ and $N = \begin{bmatrix} n & 0 & 0 \\ 0 & n & te \\ 0 & 0 & n \end{bmatrix} \in \text{nil}(V_2(L_{(s,t)}(R)))$ and $A = C + N$ is the CN decomposition of A in $V_2(L_{(s,t)}(R))$. For the inverse implication, let $r \in R$ and consider $A = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \in V_2(L_{(s,t)}(R))$. There exist $C = \begin{bmatrix} a & 0 & 0 \\ 0 & a & te \\ 0 & 0 & a \end{bmatrix} \in C(V_2(L_{(s,t)}(R)))$ and $N = \begin{bmatrix} p & 0 & 0 \\ 0 & r & tu \\ 0 & 0 & v \end{bmatrix} \in \text{nil}(V_2(L_{(s,t)}(R)))$. Then $a \in C(R)$ and $p \in \text{nil}(R)$ and $r = a + n$ is the CN decomposition of r. Hence R is a CN ring.

(2) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in CN_{(s,t)}(R)$. Set $C = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 0 \\ sc & 0 & te \\ 0 & 0 & 0 \end{bmatrix}$. By Lemma 3.1, $C \in C(L_{(s,t)}(R))$ and $N \in \text{nil}(L_{(s,t)}(R)$. $A = C + N$ is the CN decomposition of A.

(3) Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$. Let $a = x+p$, $d = y+q$ and $f = z+r$ denote the CN decompositions of a, d and f. By hypothesis $sx = sy$ and $ty = tz$. By (2) A has a CN decomposition in $L_{(s,t)}(R)$ as $A = C+N$ where $C = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \in C(L_{(s,t)}(R))$ and $N = \begin{bmatrix} p & 0 & 0 \\ sc & q & te \\ 0 & 0 & r \end{bmatrix} \in \text{nil}(L_{(s,t)}(R))$. $\square$

Corollary 3.3. *Let R be a ring. If $L_{(s,t)}(R)$ is a CN ring, then R is a CN ring.*

Proof. Assume that $L_{(s,t)}(R)$ is a CN ring and let $a \in R$ and $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in L_{(s,t)}(R)$. By hypothesis there exist $C = \begin{bmatrix} x & 0 & 0 \\ sy & z & tu \\ 0 & 0 & v \end{bmatrix} \in C(L_{(s,t)}(R))$ and $N = \begin{bmatrix} n & 0 & 0 \\ sc & m & te \\ 0 & 0 & k \end{bmatrix} \in \operatorname{nil}(L_{(s,t)}(R))$ such that $A = C + N$ where $x \in C(R)$ and $n \in \operatorname{nil}(R)$. Then $a = x + n$ is the CN decomposition of a. □

There are CN rings such that $L_{(s,t)}(R)$ need not be a CN ring.

Example 3.4. Let $R = \mathbb{Z}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \in L_{(1,1)}(R)$. Assume that $A = C + N$ is a CN decomposition of A. Since A is neither central nor nilpotent, by Lemma 3.1, we should get A had a CN decomposition as $A = C + N$ where $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C(L_{(1,1)}(R))$ and $N = \begin{bmatrix} x & 0 & 0 \\ c & y & e \\ 0 & 0 & z \end{bmatrix} \in \operatorname{nil}(L_{(1,1)}(R))$ where $\{x, y, z\} \subseteq \operatorname{nil}(\mathbb{Z})$. This leads us a contradiction in $\mathbb{Z}$.

Proposition 3.5. *R is CN ring if and only if so is $L_{(0,0)}(R)$.*

Proof. Note that $L_{(0,0)}(R)$ is isomorphic to the ring $R \times R \times R$. By Proposition 2.16, $\prod_{i \in I} R_i$ is a CN ring if and only if each R_i is a CN ring for each $i \in I$. □

The rings $H_{(s,t)}(R)$ **:** Let R be a ring and s, t be in the center of R. Let

$$H_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}.$$

Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$. Note that any element A of $H_{(s,t)}(R)$ has the form $\begin{bmatrix} sc+te+f & 0 & 0 \\ c & te+f & e \\ 0 & 0 & f \end{bmatrix}$.

Lemma 3.6. *Let R be a ring, and let s, t be in the center of R. Then the set of all nilpotent elements of $H_{(s,t)}(R)$ is*

$$nil(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid a, d, f \in nil(R), c, e \in R \right\}.$$

Proof. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in \mathrm{nil}(H_{(s,t)}(R))$. There exists a positive integer n such that $A^n = 0$. Then $a^n = d^n = f^n = 0$. Conversely assume that $a^n = 0$, $d^m = 0$ and $f^k = 0$ for some positive integers n, m, k. Let $p = max\{n, m, k\}$. Then $A^{2p} = 0$. □

Lemma 3.7. *Let R be a ring, and let s and t be central invertible in R. Then*

$$C(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid c, e, f \in C(R) \right\}.$$

Proof. [4, Lemma 3.1]. □

Theorem 3.8. *Let R be a ring. R is a CN ring if and only if $H_{(s,t)}(R)$ is a CN ring.*

Proof. Assume that R is a CN ring. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in (H_{(s,t)}(R))$. Then $a = c_1 + n_1$, $d = c_2 + n_2$, $f = c_3 + n_3$, $c = c_4 + n_4$, $e = c_5 + n_5$ with $\{c_1, c_2, c_3, c_4, c_5\} \subseteq C(R)$, $\{n_1, n_2, n_3, n_4, n_5\} \subseteq \mathrm{nil}(R)$. Let $c_1 - c_2 = sc_4$, $c_2 - c_3 = tc_5$, $n_1 - n_2 = sn_4$ and $n_2 - n_3 = tn_5$ and $C = \begin{bmatrix} c_1 & 0 & 0 \\ c_4 & c_2 & c_5 \\ 0 & 0 & c_3 \end{bmatrix}$ and

$N = \begin{bmatrix} n_1 & 0 & 0 \\ n_4 & n_2 & n_5 \\ 0 & 0 & n_3 \end{bmatrix}$. By Lemma 3.7, $C \in C(H_{(s,t)}(R))$ and by Lemma 3.6, $N \in \mathrm{nil}(H_{(s,t)}(R))$. Then $A = C + N$ is the CN decomposition of A.

Conversely, suppose that $H_{(s,t)}(R)$ is a CN ring. Let $a \in R$. Then $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in (H_{(s,t)}(R))$ and it has a CN decomposition $A = C + N$ where $C = \begin{bmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{bmatrix} \in C(H_{(s,t)}(R))$ with $\{y, u, v\} \subseteq C(R)$ and $N = \begin{bmatrix} n_1 & 0 & 0 \\ n_2 & n_3 & n_4 \\ 0 & 0 & n_5 \end{bmatrix} \in \mathrm{nil}(H_{(s,t)}(R))$ with $\{n_1, n_3, n_5\} \subseteq \mathrm{nil}(R)$. Then $a = x + n_1$ is a CN decomposition of a. □

Proposition 3.9. *Uniquely nil clean rings, uniquely strongly nil clean rings, strongly nil *-clean rings are CN.*

Proof. These classes of rings are abelian. Assume that R is uniquely nil clean ring. Let e be an idempotent in R. For any $r \in R$, $e + (re - ere)$ can be written in two ways as a sum of an idempotent and a nilpotent as $e+(re-ere) = (e+(re-ere))+0 = e+(re-ere)$. Then $e = e+(re-ere)$ and $er - ere = 0$. Similarly, $e+(er-ere) = (e+(re-ere))+0 = e+(er-ere)$. Then $0 = re - ere = re - ere$. Hence e is central. □

The converse of this result is not true.

Example 3.10. *The ring $H_{(0,0)}(\mathbb{Z})$ is CN but not uniquely nil clean.*

Proof. By Theorem 3.8, $H_{(0,0)}(\mathbb{Z})$ is CN. Note that for $n \in \mathbb{Z}$ has a uniquely nil clean decomposition if and only if $n = 0$ or $n = 1$. Let $A = \begin{bmatrix} a & 0 & 0 \\ c & a & e \\ 0 & 0 & a \end{bmatrix} \in H_{(0,0)}(R)$ with $a \notin \{0, 1\}$. Assume that A has a uniquely nil clean decomposition. There exist unique $E^2 = E = \begin{bmatrix} x & 0 & 0 \\ y & x & u \\ 0 & 0 & x \end{bmatrix} \in H_{(0,0)}(R)$ and

$N = \begin{bmatrix} g & 0 & 0 \\ h & g & l \\ 0 & 0 & g \end{bmatrix} \in N(H_{(0,0)}(R))$ such that $A = E + N$. Then A has a uniquely nil clean decomposition. So $a = x + g$ has a CN decomposition. This is not the case for $a \in \mathbb{Z}$. Hence $H_{(0,0)}(\mathbb{Z})$ is not uniquely nil clean. ☐

Generalized matrix rings: Let R be ring and s a central element of R. Then $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ becomes a ring denoted by $K_s(R)$ with addition defined componentwise and with multiplication defined in [6] by

$$\begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{bmatrix}.$$

In [6], $K_s(R)$ is called a *generalized matrix ring over R.*

Lemma 3.11. *Let R be a commutative ring. Then the following hold.*

(1) $nil(K_0(R)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_0(R) \mid \{a, d\} \subseteq nil(R) \right\}$.

(2) $C(K_0(R))$ *consists of all scalar matrices.*

Proof. (1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{nil}(K_0(R))$. Then $A^2 = \begin{bmatrix} a^2 & b(a+d) \\ c(a+d) & d^2 \end{bmatrix}$, ...,

..., $A^{2^n} = \begin{bmatrix} a^{2^n} & \Sigma_{i=1}^n b(a^{2^{i-1}} + d^{2^{i-1}}) \\ \Sigma_{i=1}^n c(a^{2^{i-1}} + d^{2^{i-1}}) & d^{2^n} \end{bmatrix}$. Hence $A \in \mathrm{nil}(K_0(R))$ if and only if $\{a, d\} \subseteq \mathrm{nil}(R)$. ☐

Lemma 3.12. *Let R be ring. Then R is a CN ring if and only if* $D_n(K_0(R))$ *is a CN ring.*

Proof. Necessity: We assume that $n = 2$. Let $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in D_2((K_0(R)))$. By assumption $a = c_1 + n_1$ where $c_1 \in C(R)$ and $n_1 \in \mathrm{nil}(R)$. Let $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} \in C(D_2(K_0(R)))$ and $N = \begin{bmatrix} n_1 & b \\ 0 & n_1 \end{bmatrix} \in \mathrm{nil}D_2((K_0(R)))$. $A = C+N$ is the CN decomposition of A.

Sufficiency: Let $a \in R$. Then $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in D_2((K_0(R)))$ has a CN decomposition $A = C + N$ with $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} \in C(D_2((K_0(R))))$ and $N = \begin{bmatrix} n_1 & b_1 \\ 0 & n_1 \end{bmatrix} \in \text{nil}(D_2((K_0(R))))$ where $c_1 \in C(R)$ and $n_1 \in \text{nil}(R)$. By comparing components of matrices we get $a = c_1 + n_1$. It is a CN decomposition of a. $\square$

Note that $K_0(R)$ need not be a CN ring.

Example 3.13. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K_0(\mathbb{Z})$ have a CN decomposition as $A = C + N$ where $C \in C(K_0(\mathbb{Z}))$ and $N \in \text{nil}K_0(\mathbb{Z}))$. Then we should have $C = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ and $N = \begin{bmatrix} 1-x & 0 \\ 0 & -x \end{bmatrix}$. These imply $x = 1$ or x is nilpotent. A contradiction.

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SOME NEW MATHEMATICAL METHODS ON QUANTUM THEORY

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Abstract

First, iso-mathematics and its wide applications constitute iso-science, which includes physics, quantum theory, chemistry, energy and cosmology, etc. Second, high dimensions of mathematics and physics, and basic equation of energy-momentum representation are researched. Third, the nonlinear quantum theory is discussed. Fourth, possible unified equation and theory of fermions and bosons are proposed. Fifth, mathematical base of the extensive quantum theories is studied. Finally, some possible developments of quantum equations are searched.

Key words: quantum theory, mathematics, iso-science, equation, unification.

I. Iso-Mathematics and Its Applications

Physics and mathematics are always closely related and promote each other. From 1978 Santilli discovered the new Iso-Mathematics, which includes isonumbers, isorepresentation [1-3], etc. In Iso-Mathematics, isonumbers and genonumbers of dimensions 1, 2, 4, 8, and their isoduals and pseudoduals may extend to “hidden numbers” of dimension 3, 5, 6, 7 [2]. They are applied for many regions [3,4].

In 1998 Santilli shown that the objections against the Einstein-Podolsky-Rosen (EPR) argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. Now Santilli reviewed, upgraded and specialized the basic mathematical, physical and chemical methods for the study of the EPR prediction that quantum mechanics is not a complete theory. This includes basic methods [5], apparent proof of the EPR argument [6], and examples and applications, in which the validity of the EPR final statement is the effect that the wave function of quantum mechanics does not provide a complete description of the physical reality. The axiom-preserving “completion” of the quantum mechanical wave function due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature [7]. Moreover, Santilli studied the classical determinism of EPR prediction by isomathematics [8].

Santilli studied generalization of Heisenberg’s uncertainty principle for strong interactions [9]. Santilli searched isorepresentation of the Lie-isotopic SU(2) algebra with application to nuclear physics and local realism [10], and studied the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen [8]. Santilli discussed foundation of theoretical mechanics [11,12], because of contact, thus continuous and instantaneous character, by therefore voiding the need for superluminal communications.

The non-linear, non-local and non-potential character of the assumed interactions render them ideally suited for their representation via the isotopic (i.e. axiom-preserving branch of hadronic mechanics [13-15]), comprising iso-mathematics and iso-mechanics, which are characterized by the isotopy of the

universal enveloping associative algebra ξ of quantum mechanical Hermitian operators A, B, ... with isoproduct, and studied Einstein-Podolsky-Rosen prediction [5-7].

Santilli first recalled the 1935 historical view by A. Einstein, B. Podolsky and N. Rosen according to which "Quantum mechanics is not a complete theory" (EPR argument), because of the inability by quantum mechanics to provide a quantitative representation of the interactions occurring in particle entanglements. They then shown, apparently for the first time, that the completion of quantum entanglements into the covering EPR entanglements formulated according to hadronic mechanics provides a quantitative representation of the interactions occurring in particle entanglements by assuming that their continuous and instantaneous communications at a distance are due to the overlapping of the wave packets of particles, and therefore avoiding superluminal communications. According to this view, entanglement interactions result to be nonlinear, non-local and not derivable from a potential, and are represented via Bohm's variable λ hidden in the quantum mechanical associative product of Hermitean operators AB=A×B via explicit and concrete, axiom-preserving realizations $A\hat{\times}B = A\lambda B$, $I\hat{\times}A = A\hat{\times}I = A$ with ensuing non-unitary structure, multiplicative unit $U/U^{+} = \hat{I} = 1/\lambda$, inapplicability' of Bell's inequalities and consequential validity of Bohm's hidden variables. We finally introduce, also apparently for the first time, the completion of quantum computers into the broader EPR computers characterizing a collection of extended electronic components under continuous entanglements, and show their apparent faster computation, better cybersecurity and improved energy efficiency [16].

More explicitly, the quantum mechanical equation for two interacting particles with coordinates r_k (k=1,2) on a Hilbert space H over the field C of complex numbers is given by the familiar Schrodinger equation [16]:

$$[\sum_{k=1,2}\frac{1}{2m_k}(p_k p_k + V(r))]\psi(r) = E\psi(r). \qquad (1)$$

By recalling the basic expression of the isolinear iso-momentum characterized by the completion of the local Newton-Leibnitz differential calculus into the non-local iso-differential calculus [20]:

$$i\hat{I}\frac{\partial}{\partial t}\psi(r) = i\frac{1}{\lambda}\frac{\partial}{\partial(r/\lambda)}\psi(r/\lambda). \qquad (2)$$

The non-relativistic version of the EPR entanglement is characterized by the iso-Schrodinger equation.

The new entanglement interaction verifies, by conception and construction, the abstract axioms of relativistic quantum mechanics although realized via the indicated universal iso-associative envelope [17,18].

Santilli researched invariant Lie-isotopic and Lie-admissible formulation of quantum deformations [19], and new conception of living organisms and its representation via Lie-admissible Hv-hyperstructures [20]. Santilli proposed foundations of hadronic chemistry and applied to new clean energies and fuels [21]. Santilli and Shillady discussed a new isochemical model of the hydrogen molecule [22]. Faster computations, since all values of Bohm hidden variable λ are very small according to all available fits of experimental data, with ensuing rapid convergence of iso-perturbative series. As a confirmation of this expectation, Santilli recalled the achievement via iso-mathematics and iso-chemistry of the first known attractive force between the identical electrons of valence coupling (see Chapter 4 of [21]), resulting in a strong valence bond that allowed the first known numerically exact representation of the experimental data for the hydrogen [22] and water [23] molecules with iso-perturbative calculations at least one thousand times faster than their conventional chemical counterparts.

Clear experimental evidence in various fields of deviations of physical reality from quantum predictions in favor of exact representations via hadronic mechanics, including deviations in nuclear physics [24]; electrodynamics [25,26]; nuclear physics [24]; condensed matter physics; heavy ion physics; time dilation for composite particles [27]; Bose-Einstein correlation [28]; cosmology [29]; and other fields. The violation of causality may explain the lack of achievement to date of controlled nuclear fusion.

Santilli proposed a unified form of possible generalizations of Heisenberg uncertainty principle for strong interactions [24]. Hadronic mechanics (HM) [25,26] has generalized the uncertainty relations in the only non-trivial way.

In 2023 Santilli reviewed and updated that the insufficiencies of quantum mechanics in nuclear physics; the completion of quantum mechanics into the axiom-preserving, Lie-isotopic branch of hadronic mechanics for the invariant representation of extended protons and neutrons under potential and contact/non-potential interactions; the exact hadronic representation of all characteristics of the neutron in its synthesis from the proton and the electron at the non-relativistic and relativistic levels; the completions of Bell's inequalities with ensuing iso-deterministic principle for strong interactions. Santilli then presented the apparent resolution of the historical objections against the reduction of all stable matter in the universe to protons and electrons and point

out a number of open problems whose treatment is beyond the capabilities of quantum mechanics, such as: the cosmological implications of the missing energy in the neutron synthesis, the prediction of negatively charged pseudo-protons, and the possible recycling of radioactive nuclear waste by nuclear power plants via their stimulated decay [30].

In a word, iso-mathematics and its wide applications constitute a new iso-science, which includes physics, quantum theory, chemistry, energy, cosmology and so on [1-30]. Further, in this paper we research some new developments of quantum theory from high dimensions of mathematics and physics, and the nonlinear quantum theory to possible unified equation and theory of fermions and bosons, etc.

2. Some Known Mathematical Methods in Quantum

In mathematical method in quantum physics, Hamiltonians may apply to classical mechanics and statistics, quantum mechanics and so on. In mathematics Hamiltonian is an operator in fact, and its extension (moment map, etc.) may apply to dynamical system, differential equation, Lie group, symplectic geometry, etc.

Some mathematical methods on quantum theory are always the important concern for physicists. Recently, Nakahira focused on identifying sudden changes in weak signals transmitted by electromagnetic or gravitational waves, and demonstrated quantum change points for Hamiltonians [31]. Carollo discussed non-Gaussian dynamics of quantum fluctuations and mean-field limit in open quantum central spin systems [32]. Jalali-mola, et al., searched topological Bogoliubov quasiparticles from Bose-Einstein Condensate in a flat band system [33].

In the standard quantum theory, the causal order of occurrence between events is prescribed, and must be definite. This has been maintained in all conventional scenarios of operation for quantum batteries. Zhu, et al., studied to allow the charging of quantum batteries in an indefinite causal order (ICO), and proposed a nonunitary dynamics-based charging protocol and experimentally investigate this using a photonic quantum switch. Their results demonstrate that both the amount of energy charged and the thermal efficiency can be boosted simultaneously. They found that ICO protocol can outperform the conventional protocols and gives rise to the anomalous inverse interaction effect, and provide new insights into ICO and its potential applications [34]. The quantum geometry has significant consequences in determining transport and optical properties in quantum materials. Kaplan, et al., used a semiclassical

formalism coupled with perturbative corrections unifying the nonlinear anomalous Hall effect and nonreciprocal magnetoresistance (longitudinal resistance) from the quantum geometry. They demonstrated the coexistence of a nonlinear anomalous Hall effect and nonreciprocal magnetoresistance in films of the doped antiferromagnetic topological insulator MnBi2Te4, and indicated that both longitudinal and transverse nonlinear transport provide a sensitive probe of the quantum geometry in solids [35]. Faugno, et al., investigated a discrete nonlinear Schrödinger equation with dynamical, density-difference-dependent gauge fields. They found a ground-state transition from a plane wave condensate to a localized soliton state as the gauge coupling is varied, and found a regime in which the condensate and soliton are both stable, and identified an emergent chiral symmetry, which leads to the existence of a symmetry-protected zero-energy edge mode. The emergent chiral symmetry relates low and high energy solitons. These states indicate that the interaction acts both repulsively and attractively [36].

3. High Dimensions and Basic Equation of Energy-Momentum Representation

The space-dimension of mathematics and physics can be extended to higher n-dimensions geometry [37]. It may be Hilbert space and corresponding quantum mechanics, and superstring. In mathematics, physics and many regions the field theory is all a very important problem. When the space-dimension is extended to n, fractal and complex, and various number-systems, the field theory and its formulas may be correspondingly extended. In these cases, Gauss's theorem and Stokes's theorem, and corresponding extensions on gradient, divergence and curl are searched [38]. Further, they are combined each other, and form multiple combinations, such as the scalar-tensor fields, the scalar-spinor fields, the vector-spinor fields, etc. These fields can be applied to physics, biology, earthquakes and social science, etc. Field theory has been widely applied in many regions of natural and social sciences, and its any development will necessarily inspire and apply to more aspects [39].

Based on the non-commutation

$$[A,B]=AB-BA=\eta\neq 0, \qquad (3)$$

of matrix, group, tensor and so on, we proposed the mathematical quantum theory. General matrix and quaternion are non-commutation, but special matrix may be commutation. If η is imaginary number, it will correspond to the extensive quantum theory. If η is real number, it will be development of quantum theory. Moreover, η may be complex number, etc. We introduced a

similar wave function and corresponding operators, various similar quantum results are derived. Further, we discussed its physical meaning and various applications. Based the general matrix we researched mathematics of unified gravitational and electromagnetic fields, and discussed the space-time equations and the simplest unifying quantum theory and general relativity. This can combine general discrete mathematics [40].

It is known that momentum-energy operators in quantum mechanics are:

$$p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}, E = i\hbar\frac{\partial}{\partial t}. \qquad (4)$$

The four-dimensional equations are:

$$p_\mu\psi = -i\hbar\frac{\partial}{\partial x_\mu}\psi. \qquad (5)$$

In quantum mechanics the time-space operators of energy-momentum representation in quantum mechanics are [41,426]:

$$x_\mu\psi = i\hbar\frac{\partial}{\partial p_\mu}\psi. \qquad (6)$$

This includes the quantum equation in energy representation:

$$T\psi = -i\hbar\frac{\partial}{\partial E}\psi. \qquad (7)$$

And the space operator equation [43]:

$$r\psi = i\hbar\frac{\partial}{\partial p}\psi. \qquad (8)$$

Based on Eq.(6) we discussed some applications, in particular, the lifetime formulas [43].

Eq.(7) may extensive to:

$$i\hbar\frac{\partial}{\partial E}\psi = -(T+V)\psi. \qquad (9)$$

This may obtain the lifetime formulas. When potential is rotation or at spherical coordinates, we may obtain $\frac{l(l+1)}{r^2}$. Such the equation with the spherical potential in bound states is:

$$\frac{d^2}{dr^2}\psi - \frac{l(l+1)}{r^2}\psi + \frac{2m}{\hbar^2}[E - V(r)]\psi = 0. \qquad (10)$$

The total energy is:

$$E = E_0 + An + Bl(l+1). \qquad (11)$$

This is similar to diatomic molecules.

Eq.(10) corresponds to equation in the spherical potential:

$$[\frac{d^2}{dp^2}-\frac{l(l+1)}{p^2}+\frac{E-V}{\hbar^2}]\psi=0. \qquad (12)$$

$$\frac{d^2}{dE^2}\psi-\frac{l(l+1)}{E^2}\psi+[T(E)-V]\psi=0. \qquad (13)$$

This adds the harmonic oscillator and the anharmonic oscillator $V=aE^2+bE^3$, and $\tau=h/E$. In momentum representation, Eq.(15) becomes to:

$$[\frac{d^2}{dp^2}-\frac{l(l+1)}{p^2}+\frac{1}{\hbar^2}(E-ap^2-bp^3)]\psi=0. (14)$$

Planck formula was originally based on a series of harmonic oscillators. This should obtain different quantum lifetime. From the time equation (7)(9), we obtained some lifetime formulas [44-46], and they agree better with the experimental data [47].

Further, we proposed some operator equations of general relativity and special relativity, for example:

$$g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi+\frac{s^2}{\hbar^2}\psi=0. \qquad (15)$$

This is the form of combining quantum mechanics and general relativity, special relativity, and is the simplest unity between relativity and quantum, and corresponds to the extensive quantum theory [48-51]. It may overcome the singularity problem in general relativity, and is the combination and unification on quantum mechanics and general relativity [43].

4. Nonlinear Quantum Theory

The nonlinear approach of quantum mechanics is continuously an important direction. We propose mathematically the basic nonlinear operators [44,52]:

$$p_\mu=-i\hbar(F\frac{\partial}{\partial x_\mu}+i\Gamma_\mu), \qquad (16)$$

so Klein-Gordon equation and Dirac equations turn out to be

$$(F^2\partial_\mu^2+\Gamma_\mu^2-m^2)\varphi=-J. \qquad (17)$$

Dirac equations are:

$$\gamma_\mu(F\partial_\mu+i\Gamma_\mu)\psi+\mu\psi=j. \quad (18)$$

We obtained the corresponding Heisenberg equation. Then the present applied superposition principle is developed to the general nonlinear form. The quantum commutation and anticommutation belong to F and Γ_μ. This theory may include the renormalization, which is the correction of Feynman rules of curved closed loops. We think the interaction equations must be nonlinear. Many theories, models and phenomena are all nonlinear, for instance, soliton, nonabelian gauge field, and the bag model, etc [44,52]. The superluminal entangled state, which relates the nonlocal quantum teleportation and nonlinearity, should be a new fifth interaction. The relations among nonlinear theory and electroweak unified theory, and QCD, and CP nonconservation, etc., are expounded. Some known and possible tests are discussed [52].

Based on a hypothesis on duality exists probably only under interactions, so its test is that duality not exists when there are not interactions. Further, some nonlinear quantum equations and wave equations have soliton solutions, which correspond to particles. This is mathematical description of wave-particle duality, and whose condition is also interactions and nonlinearity. From the soliton-chaos double solutions of nonlinear equations, we proposed the field-particle duality, and field (wave)-quantum-chaos ternary. If we develop mathematical form, more multiplicity will be obtained. In the quantum entangled states, velocities of different phases obey possibly general Lorentz transformation (GLT) [53].

We proposed that a trefoil knot of left hand and a trefoil knot of right hand is possibly similar to two spins s=1/2 and -1/2 of three quarks (nucleons p and n, etc). Two topological structures should differ in their energy levels in the magnetic field. Basic mesons of two quark form a line, and s=0. This model is first applied to proton and neutron, then it may be extended to other particles [54].

5. Possible Unified Theory of Fermions and Bosons

It is known that fermion-pairs can form bosons. Further, any bosons can be composed of fermion-pairs. Therefore, fermions are more fundamental matter units, and bosons and fields can be composed of fermion-pairs. Such as p, e constitute hydrogen atoms, quark and antiquark pairs form mesons.

The time-independent Schrödinger equation is:

$$\Delta\psi+\frac{2m}{\hbar^2}(E-V)\psi=0. \tag{19}$$

Let $\frac{2m}{\hbar^2}(E-V)=k^2$, Eq.(19) becomes to the Helmholtz equation, i.e., wave equation:

$$\nabla^2\psi+k^2\psi=0. \quad (20)$$

For the 3-dimensional central potential, let $\psi(r)=Y_{lm}(\Omega)u_{n,l}(r)/r$, Schrödinger equation is simplified to:

$$\{\frac{d^2}{dr^2}+\frac{2m}{\hbar^2}[E(n,l)-V(r)]-\frac{l(l+1)}{r^2}\}u_{n,l}(r)=0. \quad (21)$$

The angular momentum quantum number l naturally produces a rotational potential.

The spin as part of the total angular momentum shows as the electron spin s is split to 2 in the magnetic field, which should develop Pauli equation, and in Eq.(20):

$$k^2=\frac{2m}{\hbar^2}E\pm\frac{2e}{c\hbar}(l+\frac{1}{2}). \quad (22)$$

For different the angular momentum M and potential V substitution into the same k^2, the separation variables are still the same equation.

In spherical coordinates, by general method let $\psi=R(r)Y(\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+[x^2-(j+\frac{1}{2})^2]y=0. \quad (23)$$

It is j+(1/2) order Bessel equation, which is also discussed by Weinberg [55].

In column coordinates, by general method let $\psi=R(\rho)Z(z)\Phi(\varphi)$.

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+[x^2-m^2]R=0. \quad (24)$$

R(x) is m order Bessel equation. If quantized j+(1/2) and m are all spins, Eqs.(23) and (24) will correspond to fermions and bosons, respectively.

We searched possible unified equations of fermions and bosons, both correspond to j+(1/2) order Bessel equation in spherical coordinates, and m order Bessel equation in column coordinates [56]. Further, this is corresponding relations between Bessel equation and spin:

spherical coordinate	j	fermions	Dirac equation	FD statistics
column coordinate	m	bosons	K-G equation	BE statistics, and BEC

In quark model both correspond to baryons with three quarks and mesons with quark-pair, which should be at column coordinates.

If both are related with topology, so fermions in the spherical coordinates are independent of each other, and corresponds to Pauli exclusion principle; bosons in the column coordinates are open, which may have multiple particles.

It is known that Kerr solution (1963) and Kerr-Newman solution (1965) have the column symmetry. Boyer-Lindquist (1967) and Carter (1968) completed Kerr solution and the maximum expansion of Kerr-Newman solution. This should correspond to my plane solution of general relativity [57]. If this is related to bosons, so it will be unstable. For evolution, it corresponds to the DNA. We should research the relations of DNA and bosons.

In general relativity, Schwarzschild solution and Reissner-Nordstrom solution have the spherical symmetry. It is relation to fermions. Except photon (s=1), only three fermions p, e, v are stable.

The Wheeler-De Witt functional differential equations with infinite degrees of freedom can not almost be solved [58]. The covariat quantization eventually abandoned the point particle model, and developed into string theory. The loop quantum gravity is found along the canonical quantization path.

Einstein thought that space and others all originate from the field, which corresponds to the bosons. While bosons can be produced by fermions, and formed in pairs (as photon γ=vv). Most basic graviton (spin s=2), photon γ(s=1), for strong and weak interactions (except π and K, s=0) gluon and W-Z are all s=1, which should be basis of field theory for the great unification of strong and weak electromagnetic forces.

Strong and weak interactions as short-range should be unified. Except different action ranges their main character is: strong interactions are attraction each other, and weak interactions are mutual repulsion and derive decay. We proposed a new method on their unification, whose coupling constants are negative and positive, respectively. Further, we researched a figure on the unification of the four basic interactions in three-dimensional space. So far the high energy experiments in the past sixty years have shown that the smallest mass fermions are proton, electron, neutrino and photon, which form the simplest model of particles. These fermions seem to be inseparable truth "atoms" (elements), because further experiments derive particles with bigger mass. They correspond to four interactions, and are also only stable particles. The final simplest theory is based on leptons (e-ν_e) and nucleons (p-n) or (u-d) in quark model with SU(2) symmetry and corresponding Yang-Mills field. Other particles and quark-lepton are their excited states. We discussed the simplest interactions and simplified QCD, and some possibly developed

directions of particle physics, for example, violation of basic principles, and precision and systematization of the simplest model, etc [59,60].

After three generations of quarks-leptons and Higgs boson have been confirmed, we researched some possible development directions of particle physics [61]. They include that particles of three generations are extended, the hadronic theory must precision quantification such as heavy flavor hadrons and the lifetime formula, etc., some basic principles (Pauli Exclusion Principle, duality, uncertainty principle, etc.) are probably violated under some cases. Santillli, Penrose, et al., proposed some new mathematical methods. Moreover, particle astrophysics, the negative matter as unified dark matter and dark energy [62-67] are searched.

Further, complex number, quaternion (whose unit constitutes SU(2) group), octonion [68], matrix and other number field or number system may be extended [69]. They may be applied to quantum theory and entanglement, communication [70-72], etc. We should research the entangled field is what field? Probably this is scalar, or vectors, tensor, spinor field, etc. It seems to be dual fields, inductive fields resonant to each other, and is the wave-field with the same frequency.

6. Mathematical Base of The Extensive Quantum Theories

According to Feynman's idea [73] and based on a new form of the Titius-Bode law [48,49]:

$$r_n = an^2, \qquad (25)$$

we developed a similar theory with the Bohr atom model, and obtain the quantum constants $H = (aGM_O)^{1/2}$ of the solar system and corresponding Schrödinger equation [48,49]:

$$iH\frac{\partial\psi}{\partial t} = -H^2\nabla^2\psi + (U - Q)\psi. \qquad (26)$$

Further, we proposed the extensive quantum theory in which the formulations are the same with the quantum mechanics and only quantum constant h and corresponding basic quantum elements are different [48-51].

The extensive quantum theory is based on the probability theory, and has statistical properties. It is related with fluid dynamics and its equations, and may be combined with Schrödinger equation, KG equation and Dirac equations. This is combined with the probability equation, and corresponds to synergetics. The system of natural units (c=h=1) is namely a unified extensive form.

The extensive quantum theory has wave, interference, superposition, entanglement, coherence and interaction, etc. Coherence, entanglement, and interactions of macromolecules may produce biology, life, A-T, G-C, and DNA double helix structures, etc.

Further, we researched, the extensive special relativity [44,74] and the extensive general relativity [75], etc. These theories representations are the same with those original theories, only the basic constants can be different. Mathematical bases of various extensive theories are all the isomorphism, and may be based on the system theory, symmetry, fractal, etc. The scaling invariance is mathematical basis of various extensive theories. This should have the corresponding renormalization group equation, in the original equation, , m, etc., transform to c, h, etc.

7. Some Possible Developments of Quantum Theory

The essence of the renormalization is to eliminate the infinite background, i.e., related to the vacuum and the infinite potential. These and the renormalization are not required in the gravitational field.

It is known that baryon is Dirac equation, and meson is Klein-Gordon equation. Both are approximately Schrödinger equation. This may obtain $r_n = an^2$, and corresponding equations may derive $p_n = bn^2$, and $E_n = cn^2 = \frac{p^2}{2m}$, $p = Dn$, etc.

Weinberg proposed that we should take it seriously and find a more satisfying other possible theory, and quantum mechanics is only a good approximation of it [55].

We discussed possible 16 types developments of quantum equations [56].

Life changes with energy. There are vibrations, such as resonant or nonresonant, and rotation. It can be used to study life, its spectrum and quantized.

Space-time has an uncertainty relation, and is quantization. Combined with the logical structure of quantum mechanics [76], it may have statistics. p and r conjugate, E and t conjugate, M_z and φ conjugate. Three are symmetry, and form uncertainty relations each other.

This method can be generalized to the mechanical wave theory [77], and obtain the conjugate equation, and the mass-lifetime formula, etc.

The three-generation quark equations may be unified. Only n, l and Y, Q are different, except u, d, the other I all is 0. There have relations with P, C, PC, and PCT.

The relativity equation is:

$$s^2 = T^2 - R^2, T^2 = s^2 + R^2. \qquad (27)$$

$$T^2\psi = -\hbar^2 \frac{\partial^2}{\partial E^2}\psi = (s^2 + R^2)\psi. \qquad (28)$$

$$\frac{\partial^2}{\partial p_\mu^2}\psi = [\frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial E^2}]\psi = \frac{s^2}{\hbar^2}\psi. \qquad (29)$$

It is similar Klein-Gordon equation. The similar Dirac equations are:

$$i\hbar\frac{\partial}{\partial E}\psi = (\alpha s + \beta R)\psi. \qquad (30)$$

$$\gamma_\mu \frac{\partial}{\partial p_\mu}\psi = s\psi. \qquad (31)$$

It is replaced by various space-time metric, and can obtain various equations of similar quantum mechanics, such as for the vacuum $s^2 = 0$.

We should research equation and so on for anyon.

It is known that the charged Dirac equations become non-relativistic, so can obtain spin. By similar way some effect should be obtained. But this must first consider the change in space-time in the electromagnetic field, as for Reissner-Nordstrom metric

$$g_{00}c^2dt^2 = (1 - \frac{2m}{r} + \frac{4\pi GQ^2}{c^4r^2})c^2dt^2. \qquad (32)$$

The equation with charge is:

$$(\gamma_\mu \frac{\partial}{\partial p_\mu} - s + AQ^n)\psi = 0. \qquad (33)$$

This should combine the electromagnetic general relativity [78].

When time has a direction, the corresponding equations may be developed. At this time the time may be the vector, and corresponds to the 3 dimension time [79]:

$$-\hbar^2\Delta\psi = \vec{t}\,\psi. \qquad (34)$$

It can introduce the spherical coordinates, and obtain $\frac{l(l+1)}{r^2}$.

This follows that space-time changes with energy, and consistent with

general relativity. For general relativity, $s^2 = g_{\mu\nu}x_\mu x_\nu$,

$$x_\mu\psi = i\hbar\frac{\partial}{\partial p_\mu}\psi = (s^2 / g_{\mu\nu}x_\nu)\psi. \qquad (35)$$

General case is:

$$s^2\psi = g_{\mu\nu}x_\mu x_\nu\psi = -\hbar^2 g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi. \qquad (36)$$

There is the electromagnetic field, it is:

$$g_{\mu\nu}x_\mu x_\nu\psi + \hbar^2 g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi + A(Q^2 + V)\psi = 0. \qquad (37)$$

For special relativity,

$$s^2\psi = [(ct)^2 - r^2]\psi = -\hbar^2[\frac{\partial^2}{\partial(E/c)^2} - \frac{\partial^2}{\partial p^2}]\psi. \qquad (38)$$

Various metrics, as Schwarzschild metric

$$(1-\frac{2m}{r})c^2dt^2. \qquad (39)$$

$$T^2\psi = (1-\frac{2m}{r})(-\hbar^2\frac{\partial^2}{\partial E^2})\psi. \qquad (40)$$

$$(1-\frac{2m}{x_\mu})x_\mu^2\psi = (x_\mu^2 - 2mx_\mu)\psi = (-\hbar^2\frac{\partial^2}{\partial p_\mu^2} - 2im\hbar\frac{\partial}{\partial p_\mu})\psi. \qquad (41)$$

This equations have the first-order and second-order differential. Reissner-Nordstrom metric is $\frac{R^2(t)}{1-kr^2}dr^2$. $g_{\mu\nu}$ correspond to E and V.

We can further unify quantum theory and general relativity. Relativity focuses on space-time, while quantum theory focuses on matter and mass-energy. This can particularly describe the quantized discrete space-time, in which dx_μ is a quantum space-time. We proposed the general equations are [43]:

$$G_{\mu\nu}\psi = \kappa T_{\mu\nu}\psi. \qquad (42)$$

This should be replaced by the operator representations of $G_{\mu\nu}$ or/and $T_{\mu\nu}$ in the energy-momentum and space-time. It can be related to the Wheeler-de Witt cosmic equations of quantum mechanics.

It can be similarly developed by method of nonlinear quantum theory

[44,52], in which F corresponds to $g_{\mu\nu}$, and Γ corresponds to potential V.

In astronomy and cosmology, it corresponds to general relativity. Such various models of cosmic variation can be obtained, as the Big Bang, inflation, the circular universe, etc.

Charged particles may combine the cosmical electrodynamics [79].

Combining extensive quantum theory [48-51], the possible wave functions must also be generalized to extensive quantum theory.

In statistical mechanics and entropy,

$$S\psi = \frac{E}{T}\psi = i\hbar\frac{1}{T}\frac{\partial}{\partial t}\psi. \qquad (43)$$

The simplest is h→h/T.

Based on the Noether theorem, energy E is related to time t: 1). E is conserved, t is uniform; E is constant, and t is also constant. 2). E change, t has direction; especially when dE/T=dS. 3). More generally, when various changes, t shows the direction. 4). E and various cycles, t also cycle, or spiral rise. 5). E slows, as does t and longevity. 6). When chaotic and nonlinear change, t is not uniform and has very critical moments.

Wendt studied systematically quantum mind and social science, and proposed that people are the walking wave functions based on the causal closure (or completeness) of physics (CCP) [80]. This research core is that quantum theory can explain the phenomena of consciousness and intentionality, and an important basis is the quantum entanglement and macroscopic entanglement. We discussed quantum sociology, whose bases are the extensive quantum theory and the social individual-wave duality [81,82]. Quantum may be extended to people, animals and plants, all the life span, evolution, etc. V (E) is a factor affecting life, time and its structure, and is external factor and external potential. Infinite deep potential trap, i.e., any life is all limited, and different people have different energy levels. It may combine Maslow's different needs.

In a word, quantum theory and its mathematics may be applied to many aspects. It will be useful and very meaning.

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ISOAFFINE CONNECTIONS AND SANTILLI'S ISORIEMANNIAN METRICS ON AN ISOMANIFOLD

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Abstract

Let M be an isomanifold. The aim of the present paper is to study isoaffine connections and Santilli's isoRiemannian metrics on the isomanifold M. In order to give global definitions of these notions we need to defined some isoalgebraic structures.

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Key Words: Santilli's isounit, isofields, isovector spaces, isotopic element, isolinear transforation, isoinner product, isobilinear form, isoaffine connection, isomanifolds and isoriemannian metric.

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1. Introduction

The isostructures have been introduced by R. M. Santilli. The isostructures can be split in two big categories. This can be represented by the following diagram:

ISOSTRUCTURES

ISOALGEBRAIC STRUCTURES ISOTOPOLOGICAL STRUCTURES

The aim of the present paper is to study isoaffine connections and Santilli's isoRiemannian metrics on an isomanifold. In order to define these notions and specially globally we need the notions of some basic isoalgebraic structures. These are analyzed as follows:

The theory of isounit is contained in the second paragraph. Since the isounit is a basic notion for the whole theory of the isoctructures for this reason we describe this theory in more details.

The third parapraph includes study of isofields, which play important role in the isotopological structures.

The isovector spaces, isolinear transformations, isobilinear forms and isoinner products are studied in the fourth paragraph

In the fifth paragraph we study the isoEuclidean spaces and as well as the isoLorentzian spaces.

The theory of isomanifolds and some other objects are considered in the sixth paragraph.

The isoaffine connections on an isomanifold are contained in the seventh paragraph.

The last paragraph contains the isoRiemannnian isomanifold and all the other topics which are related to this notion.

2. Isounit

Let V be a set which has a law of composition $\square$ with unit e. To each element $\hat{I} \in V$ we have its inverse $\hat{T} = \hat{I}^{-1}$. Let X be a set with some structures. from this set we can construct another set

$$\hat{X} = \{\hat{x} = x\,\hat{I} \;/\; x \in X\}$$

under the condition this multiplication $x\,\hat{I} = \hat{x}$ can be defined. The set $\hat{X}$ is called isotopic of X und the element $\hat{I}$ is called isounit. The lifting from X onto $\hat{X}$ is called isotopy by means of $\hat{I}$. This can be presented by the diagram

$$X \xrightarrow{\hat{I}} \hat{X}$$

The inverse element $\hat{T}$ of $\hat{I}$ is called isotopic element. Therefore in the theory of isostructures an important role plays the study of isounit.

We assume that X has a law of composition $\square$ with unit I which is fixed as independent from other factors

However the isounit $\hat{I}$ is of the same dimension of I, but with unrestricted functional dependance on the local coordinates $\{y\}$ of the element $x \in X$, their derivatives of arbitrary order $\{\dot{y}, \ddot{y}, \ldots\}$ as well as any additional quantities, that means the isounit $\hat{I}$ has the form

$$\hat{I} = \hat{I}\,(y, \dot{y}, \ddot{y}, \ldots, r_1, \ldots, r_k)$$

where $r_1, \ldots, r_k$ special quantities, which we need for the special problem.

We must note though the isotopies occur when $\hat{I}$ preserves all the topological characteristics of $\hat{I}$, which are real-valuedence, positive definiteness and nowhere-degenerancy.

It can be easily seen that from isounit $\hat{I}$ we obtain its neqative - $I = I^d$, that is, we have the mapping

$$\hat{I} \rightarrow \hat{I}^d = -\hat{I}$$

which is called isoduality providiny an antiautomorphic image of all formulations based on $\hat{I}$.

In the above we have considered the isounit $\hat{I}$ with the property that it is positive definite. However, we can study some other cases which are included in the ([4]).

3. Isofields

Let F $(\alpha, +, \cdot)$ be an ordinary field of characteristic zero with generetic elements α. It is known that on F there are the two laws of composition (+, x), that means

$$+ \; : \; F \times F \;\rightarrow\; F \,, \qquad + : (\alpha_1, \alpha_2) \rightarrow \alpha_1 + \alpha_2$$

$$\cdot \; : \; F \times F \;\rightarrow\; F \,, \qquad \cdot : (\alpha_1, \alpha_2) \rightarrow \alpha_1 \cdot \alpha_2$$

For these two laws of composition we have the properties

For the + there is an element $0 \in F$ with the property $\alpha + 0 = \alpha$, $\forall \alpha \in F$.
0 is called zero.

For the α there is an element I with the property $I \cdot \alpha = \alpha$, $\forall \alpha \in F$.
I is called unit.

Now, we lift the unit I of F onto the isounit $\hat{I}$, with $\hat{I}$ being usualy outside of the original set with the method which has been used above. Now, we costruct the isoset

$$\hat{F} = \{\hat{\alpha} = \alpha \hat{I} \;/\; \alpha \in F\}$$

PROPOSITION 3.1. *The set $\hat{F}$ can become a field, which is called isofieled associated to F by means of the isounit $\hat{I}$.*

Proof: On the set $\hat{F}$, which is the isotopy of F, we define the following two laws of composition

$$+_1 : \hat{F} \times \hat{F} \to \hat{F}, \quad +_1 : (\hat{\alpha}_1, \hat{\alpha}_2) \to \hat{\alpha}_1 +_1 \hat{\alpha}_2 \overset{\text{def}}{=} (\alpha_1 + \alpha_2)\, I$$

$$* : \hat{F} \times \hat{F} \to \hat{F}, \quad * : (\hat{\alpha}_1, \hat{\alpha}_2) \to \hat{\alpha}_1 * \hat{\alpha}_2 = \hat{\alpha}_1\, T\, \hat{\alpha}_2 = \alpha_1\, \hat{I}\, T\, \alpha_2\, \hat{I} =$$

$$= (\alpha_1 \cdot \alpha_2)\, \hat{I} = \alpha_1 \hat{\cdot} \alpha_2 .$$

These make $\hat{F}$ a field whose zero is 0. The same as in F, the unit is the isounit $\hat{I}$. The elements $\hat{\alpha} \in \hat{F}$ are called isonumbers.

COROLLARY 3.2. *The usual operations between the numbers are valid for the isonumbers, that is*

$$\alpha + \beta \to \hat{\alpha} +_1 \hat{\beta} = (\alpha + \beta)\hat{I}$$

$$\alpha \times \beta \to \hat{\alpha} * \hat{\beta} = (\alpha \hat{\cdot} \beta)$$

$$\alpha^{-1} \to \hat{\alpha}^{-1} = \alpha^{-1}\hat{I}, \quad \alpha/\beta = \gamma \to \hat{\gamma} = \gamma\hat{I}$$

$$\alpha^{1/k} \to \hat{\alpha}^{1/k} = \alpha^{1/k}\hat{I}$$

Proof: This can be easily proved by means of the construction of the isofield.

An immediate consequence of the above are the following properties of the isounit

$$\hat{I} * \hat{I} * \ldots * \hat{I} = \hat{I}, \quad \hat{I} / \hat{I} = \hat{I}, \quad \hat{I}^{1/k} = \hat{I}$$

Let $F(\alpha, I, +, \times)$ be a field. From the isounit $\hat{I}$ we can construct the isofield $\hat{F}(\hat{\alpha}, \hat{I}, +_1, *)$. We also can obtain another isofield $\hat{F}(\hat{\alpha}^d, \hat{I}^d, +_1, *^d)$ as follows:

$$\hat{I}^d = -\hat{I} \qquad \hat{\alpha}^d = \bar{\alpha}\,\hat{I}^d \qquad \hat{+}_1{}^d \quad \text{in the usual way}$$

$$*^d = x\,T^d\,x\,, \quad \text{where } T^d = -T$$

We nust notice that we have $\alpha^d = -\alpha$ for the real numbers. For the complex numbers we set $c^d = -\bar{c}$, where $\bar{c}$ is the ordinary complex number. If the number β is a quaternion, which is represented by a matrix, then $\beta^d = -\beta^{\dagger}$, where $\dagger$ is thi Hermitian conjugate.

The isofield $\hat{F}^d$ is called isodual isofield. The elements $\hat{\alpha}^d \in \hat{F}^d$ are called isodual isonumber. The isodual isosum and isodual isomultiplication are given by

$$\hat{\alpha}^d +_1 \hat{\beta}^d = (\bar{\alpha} + \bar{\beta})\,\hat{I}^d$$

$$\hat{\alpha}^d *^d \hat{\beta}^d = \hat{\alpha}^d T^d \hat{\beta}^d = -\hat{\alpha}^d\,T\,\hat{\beta}^d = (\bar{\alpha} \cdot \bar{\beta})\,\hat{I}^d$$

EXAMPLE 3.3. The isofield of isoreal numbers

$$\hat{\mathbb{R}} = \{\hat{\alpha} = a\,\hat{I} \;/\; a \in \mathbb{R}\,,\ \hat{I} \text{ isounit}\}$$

$\hat{T} = \hat{I}^{-1}$ the isotopic element. It is possible $\hat{I}$ can have the form

$$\hat{I} = \hat{I}(t, x, \dot{x} \ldots, x^{(k)})$$

The elements $\hat{\alpha}$ are called isoreal numbers.

EXAMPLE 3.4. The isofield of isocomplex numbers

$$\hat{\mathbb{C}} = \{\hat{z} = z\hat{I} \;/\; z \in \mathbb{C}\,,\ \hat{I} \text{ isounit}, T = \hat{I}^{-1}\}.$$

It is possible the isounit $\hat{I}$ to have the form

$$\hat{I} = \hat{I}(t, z, \dot{z} \ldots, z^{(k)})$$

REMARK 3.5. In a similar manner we can extend the field of quaternion numbers $\mathbb{H}$ into the isofield of isoquaternion numbers $\hat{\mathbb{H}}$.

4. Isovector Spaces

Let V be a vector space over a field F of characteristic zero. Let $\hat{F}$ be the isofield of F, which was constructed by the isounit $\hat{I}$. V can become a vector space over $\hat{F}$ as follows:

$$+ : V \times V \to V, \quad + : (X, Y) \to X + Y$$

Therefore, the addition + on V remains the same, independend of the field in which we work.

The external law $\otimes$ is defined by

$$\otimes \ : \hat{F} \times V \to V, \quad \otimes \ : (\hat{\lambda}, X) \to \hat{\lambda} \otimes X, \ \forall x \in V, \ \hat{\lambda} \in \hat{F}$$

The external law has the properties

(I) $\hat{\alpha} \otimes (\hat{\beta} \otimes X) = (\hat{\alpha} * \hat{\beta}) \otimes X$

(II) $\hat{\alpha} \otimes (X + Y) = \hat{\alpha} \otimes X + \hat{\alpha} \otimes Y$

(III) $(\hat{\alpha} +_1 \hat{\beta}) \otimes X = \hat{\alpha} \otimes X + \hat{\beta} \otimes X$

$$X \otimes \hat{T} = \hat{T} \otimes X = X$$

This vector space is denoted by $\hat{V}$ and it is called isovector space.

REMARK 4.1. In this lifting from V into $\hat{V}$ the elements of the linear space V remain unchanged. These elements of $\hat{V}$ are called isovectors. When we consider as elements of V are called vectors. From this remark we have that a base of the vector space V remains invariant. In other words it is a base of $\hat{V}$.

4.2. Isolinear transformation

Let $\hat{V}$ and $\hat{V}'$ be two isovector spaces over the isofield $\hat{F}$.

An isomap

$$\hat{f} : \hat{V} \rightarrow \hat{V}'$$

which preserves the sum and isomultiplication

$$\hat{f}\ (\hat{\alpha} \otimes X + \hat{\beta} \otimes Y) = \hat{\alpha} \otimes f(X) + \hat{\beta} \otimes f(Y)$$

$\forall\, X, Y \in \hat{V}$ and $\forall\, \hat{\alpha}, \hat{\beta} \in \hat{F}$, is called isolinear transformation.

If $\{\hat{e}_1, ..., \hat{e}_n\}$ and $\{\hat{t}_1, ..., \hat{t}_k\}$ are two bases of $\hat{V}$ and $\hat{V}'$ respectively, then we have

$$\left.\begin{aligned} \hat{f}(e_1) &= \hat{\alpha}_{11} \otimes t_1 + \hat{\alpha}_{12} \otimes t_2 + ... + \hat{\alpha}_{1k} \otimes t_k \\ \hat{f}(e_2) &= \hat{\alpha}_{21} \otimes t_1 + \hat{\alpha}_{22} \otimes t_2 + ... + \hat{\alpha}_{2k} \otimes t_k \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \hat{f}(e_n) &= \hat{\alpha}_{n1} \otimes t_1 + \hat{\alpha}_{n2} \otimes t_2 + ... + \hat{\alpha}_{nk} \otimes t_k \end{aligned}\right\} \tag{4.1}$$

From the relations (4.1) we obtain the matrix

$$B = {}^tA = \begin{pmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{21} & \cdots & \hat{\alpha}_{n1} \\ \hat{\alpha}_{12} & \hat{\alpha}_{22} & \cdots & \hat{\alpha}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{\alpha}_{1k} & \hat{\alpha}_{2k} & \cdots & \hat{\alpha}_{nk} \end{pmatrix}$$

which is called isorepresentation of the isolinear transformation $\hat{f}$ with respect to the bases

$$(\hat{e}_1, \hat{e}_2, ..., \hat{e}_n), \quad (\hat{t}_1, \hat{t}_2, ..., \hat{t}_k)$$

and $\hat{\alpha}_{ij}$, $i=1,...,n$, $j=1,...k$, are isonumbers.

4.3. Isobilinear form

Let $\hat{V}$ be an isovector space over the isofield $\hat{F}$. The isomapping

$$\hat{f} : \hat{V} \times \hat{V} \to \hat{F}$$

with the properties

$$\hat{f}\,(\hat{\alpha} \otimes X + \hat{\beta} \otimes Y, Z) = \hat{\alpha} \otimes \hat{f}\,(X, Z) +_1 \hat{\beta} \otimes \hat{f}\,(Y, Z)$$

$$\hat{f}\,(\hat{X}, \hat{\alpha} ; Y + \hat{\beta} ; Z) = \hat{\alpha}\, \hat{f}\,(\hat{X}, \hat{Y}) +_1 \hat{\beta}\, \hat{f}\,(\hat{X}, \hat{Y})$$

is called isobilinear form.

If $\{\hat{e}_1, \ldots, \hat{e}_n\}$ is a base of $\hat{V}$, then

$$F(\hat{e}_i \hat{e}_j) = \hat{\alpha}_{ij}$$

is an isonumber.

We consider the isomatrix

$$A = \begin{pmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{12} & \cdots & \hat{\alpha}_{1n} \\ \hat{\alpha}_{21} & \hat{\alpha}_{22} & \cdots & \hat{\alpha}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{\alpha}_{n1} & \hat{\alpha}_{n2} & \cdots & \hat{\alpha}_{nn} \end{pmatrix}$$

which is called isorepresentation of the isobilinear form.

4.4. Isoinner product

Let $\hat{V}$ be an isovector space over the isofield $\hat{F}$. We consider the isobilinear form $<\hat{}>$ on $\hat{V}$, that is,

$$<\hat{}> \;:\; \hat{V} \times \hat{V} \to \hat{F}, \quad <\hat{}> \;:\; (\hat{X}, \hat{Y}) \to <X \hat{,} Y> \in \hat{F}$$

with the additional properties

(I) $\qquad <X \hat{,} X> \geq \hat{O}$

(II) $\qquad <X \hat{,} Y> = <Y \hat{,} X>

$\forall$ X, Y $\in \hat{V}$ and $\hat{O}$ the isozero of the isofield $\hat{F}$ which is an ordered set.

This isobilinear form is called isoinner product.

EXAMPLE 4.5. Let $\mathbb{R}^n$ be the Cartesian vector space of dimension n. From the field of real numbers $\mathbb{R}$ using the isounit $\hat{I}$ having inverse element $\hat{T}$, we obtain the isofield $\hat{F}$ of isoreal numbers $\hat{\mathbb{R}}$. Now, we get the isovector space $\hat{\mathbb{R}}^n$, which is called Cartesian isovector space. Determine an isolinear transformation, its isorepresentation and isoinner product on $\hat{\mathbb{R}}^n$.

Solution : Let $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$ be the natural base of $\mathbb{R}^n$, which at the same time is a base of $\hat{\mathbb{R}}^n$. Each element $v \in V$ which is also an element of $\hat{V}$ is called isovector denoted by $\hat{v}$. We consider an isolinear transformation $\hat{f}$ on $\hat{V}$, that is

$$\begin{aligned}
&\hat{f} : \hat{V} \to \hat{V} \\
&\hat{f} : e_1 \to \hat{f}(e_1) = \hat{\alpha}_{11}e_1 + \hat{\alpha}_{12}e_2 + \dots + \hat{\alpha}_{1n}e_n \\
&\hat{f} : e_2 \to \hat{f}(e_2) = \hat{\alpha}_{21}e_1 + \hat{\alpha}_{22}e_2 + \dots + \hat{\alpha}_{2n}e_n \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\hat{f} : e_n \to \hat{f}(e_n) = \hat{\alpha}_{n1}e_1 + \hat{\alpha}_{n2}e_2 + \dots + \hat{\alpha}_{nn}e_n
\end{aligned}$$

where $\hat{\alpha}_{ij} \in \hat{\mathbb{R}}$, i=1,...,n, j=1,...,n are isoreal numbers.

The following isomatrix

$$B = {}^tA = \begin{pmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{21} & \cdots & \hat{\alpha}_{n1} \\ \hat{\alpha}_{12} & \hat{\alpha}_{22} & \cdots & \hat{\alpha}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{\alpha}_{1n} & \hat{\alpha}_{2n} & \cdots & \hat{\alpha}_{nn} \end{pmatrix}$$

is the isorepresentation of the isolinear transformation $\hat{f}$ with respect to the base $\{e_1 = \hat{e}_1, ..., e_n = \hat{e}_n\}$.

Now, we obtain two isosectors $\hat{X}$, $\hat{Y} \in \hat{V}$ which can be written

$$\hat{X} = \hat{\alpha}_1\hat{e}_1 + \hat{\alpha}_2\hat{e}_2 + \dots + \hat{\alpha}_n\hat{e}_n, \quad \hat{Y} = \hat{\beta}_1\hat{e}_1 + \hat{\beta}_2\hat{e}_2 + \dots + \hat{\beta}_n\hat{e}_n .$$

The Euclidean isoinner product is defined by

$$<\hat{}> \; : \; \hat{\mathbb{R}}^n \times \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}$$

$$<\hat{}> \; : \; (\hat{X}, \hat{Y}) \to <\hat{X}\hat{,}\hat{Y}> = \sum_{i=1}^{n} \hat{\alpha}_i * \hat{\beta}_i$$

5. Isometric Spaces

Let $\mathbb{R}^n$ be the Cartesian space of dimension n, with natural coordinate system $x = (x_1, \dots, x_n)$. On this space we can define the Euclidean metric

$$ds_E^2 = dx_1^2 + dx_2^2 + \dots + dx_n^2$$

Therefore the pair $(\mathbb{R}^n, ds_E^2)$, which is called Euclidean space of dimension n, denoted also $(\mathbb{R}^n, ds_E^2, x)$. We also can define another metric of the form

$$ds_L^2 = dx_1^2 + \dots + dx_p^2 - dx_{p+1}^2 - \dots - dx_n^2$$

This metric is called Lorentzian of the form $(p, q = n - p)$. Hence, the pair $(\mathbb{R}^n, ds_L^2\,(p, q = n-p))$ is called Lorentzian space of type $(p, q = n-1)$, denoted also $(\mathbb{R}^n, ds_L^2, \hat{x})$.

Let $\hat{I} = (\hat{I}^i_j)$ be an isounit with isotopic element $\hat{T}$. From this we obtain the isofield $\hat{\mathbb{R}}(\hat{x}, +_1, *)$ where $\hat{x}$ is isoreal number.

From this we obtain the isoreal Cartesian space

$$\hat{\mathbb{R}}^n = \hat{\mathbb{R}}^n \times \dots \times \hat{\mathbb{R}}^n = \{ \hat{x} = (\hat{x}_1 \dots \hat{x}_n) / \hat{x}_i \in \hat{\mathbb{R}} \}.$$

On the $\mathbb{R}^n$ we have the original n-dimensional unit $I = \mathrm{diag}(1, 1, ..., 1)$. Now, in $\hat{\mathbb{R}}^n$ we have the lifting of n-dimensional unit I into $n \times n$-dimensional isounit

$$\hat{I} = (I^i_j) = \hat{T}^{-1}$$

and we have deformation of the metric ds^2_E, that means

$$ds^2_E \to \hat{ds}^2_E = T ds^2_E .$$

under the assumption that the isounit of the underlying isofield coincides with that of the isoreal Cartesian space which sometimes is called real Cartesian isospace; and use of the original local coordinates in contravariant form $\hat{x}_k = x_k$, although different coordinates in their covariant form $\hat{x}_k = \delta_{ki} \hat{x}_i$. Here we must notice that the symbol x will be used for the coordinates of conventional spaces, while $\hat{x}$ will be used for the coordinates of isospaces. When we write $\hat{ds}^2_E(x, \dot{x}, \ddot{x}, ...)$ we refer to the projection of the isometric $\hat{ds}^2_E$ in the original space $\mathbb{R}^n$. Since the deformation of the metric $ds^2_E, \to \hat{T}\, \hat{ds}^2_E$ is compensated by the inverse deformation of the unit $I \to \hat{I} = T^{-1}$. Therefore, it is easy to see that $(\hat{\mathbb{R}}^n, \hat{ds}^2_E, \hat{x})$ preserves all geometric axioms of $(\mathbb{R}^n, ds^2_E, \hat{x})$. Hence, the map

$$(\mathbb{R}^n, ds^2_E, x) \to (\hat{\mathbb{R}}^n, \hat{ds}^2_E, \hat{x})$$

is an isotopy. $(\hat{\mathbb{R}}^n, \hat{ds}^2_E, \hat{x})$ is called isoEuclidean space or Euclidean isospace.

REMARK 5.1. If in the isoreal Cartesian space $\hat{\mathbb{R}}^n$ we define the isometric

$$\hat{ds}^2_L = \hat{T}\, ds^2_L \quad \text{type } (p, q=n-p)$$

then $(\hat{\mathbb{R}}^n, \hat{ds}^2_L, \hat{x})$ is called isoLorentzian space or Lorentzian isospace. We can also have the same conclusions for $(\hat{\mathbb{R}}^n, \hat{ds}^2_L, \hat{x})$ as for $(\hat{\mathbb{R}}^n, \hat{ds}^2_E, \hat{x})$.

REMARK 5.2. We must note that the lifting

$$(\mathbb{R}^n, ds_E^2, x) \to (\hat{\mathbb{R}}^n, \hat{ds}_E^2, \hat{x})$$

has significant implications. We also have the functional dependence of the matrix elements $\hat{I}^i_j$ of the isounit $\hat{I}$ is completely unrestricted. Hence, the isometric $\hat{ds}_E^2$ can depend on the local coordinates $x = (x_1, \ldots, x_n)$, as well as their derivatives with respest to an independent variable t of arbitrary order. Therefore, we have the below liftings

$$I = \mathrm{diag}(1, 1, \ldots, 1) \to \hat{I} = (\hat{I}^i_j) = \hat{I}(t, x, \dot{x}, \ddot{x}, \ldots) = \hat{T}^{-1}$$

$$ds_E^2 = |x|^2 = x_i \delta_{ij} x_j \in \mathbb{R} \;\to\; \hat{ds}_E^2 = |\hat{x}|^2 = [\hat{x}_i \hat{ds}_E^2 (t, x, \dot{x}, \ddot{x}, \ldots) \hat{x}_j], \; \hat{I} \text{ isounit}$$

where $\hat{ds}_E^2 = \hat{T}\, ds_E^2$.

It is known that $(\mathbb{R}^n, ds_E^2, x)$ is a flat. Despite the above generalization the new real isospace

$$(\hat{\mathbb{R}}^n, \hat{ds}_E^2, \hat{x})$$

is again isoflat. Hence, the isotopy preserves the flateness. Hence, the projection

$$(\hat{\mathbb{R}}^n, \hat{ds}_E^2, \hat{x}) \to (\mathbb{R}^n, ds_E^2, x)$$

preserves the flatness.

REMARK 5.3. The isoreal space $(\hat{\mathbb{R}}^n, \hat{ds}_E^2, \hat{x})$ is a special case of isoRemannian isomanifold, which can be obtained by a Riemannian manifold by lifting the metric into isometric under the condition that some basic facts of the Riemannian metric are preserved.

6. Isomanifolds

6.1. Isoreal Cartesian Isomanifold

Let $\hat{\mathbb{R}} = \{ \hat{x} = x\hat{I} \,/\, x \in \mathbb{R} \}$ be the isofield of isoreal numbers. We consider the Cartesian product of $\hat{\mathbb{R}}$ n times, that is

$$\underbrace{\hat{\mathbb{R}} \times \hat{\mathbb{R}} \times \dots \times \hat{\mathbb{R}}}_{n} = \hat{\mathbb{R}}^n = \{\hat{x}=(\hat{x}_1, \dots, \hat{x}_n) \,/\, \hat{x}_i \in \hat{\mathbb{R}}, i=1, \dots, n\} .$$

$\hat{\mathbb{R}}^n$ is called isoreal Cartesian space. On this we can define an isovector structure, an isoaffine structure, an isotopological structure and isoorthogonal structure.

Therefore, $\hat{\mathbb{R}}^n$ with all these isostructures provided with the isomapping

$$\hat{f} : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n , \quad \hat{f} : \hat{P} : \to \hat{F}(\hat{P}) = \hat{P} , \quad \forall \hat{P} \in \hat{\mathbb{R}}^n$$

is called isoreal Cartesian isomanifold.

6.2. An isochart

Let M be a Haussdorff space. A pair $(V_\alpha, \varphi_\alpha)$, where $V_\alpha \subseteq M$, is called an isochart on M, if φ_α is an isomapping:

$$\varphi_\alpha : V_\alpha \to \varphi_\alpha(V_\alpha) \subseteq \hat{\mathbb{R}}^n$$

where $\varphi_\alpha(V_\alpha)$ is an open subset of $\hat{\mathbb{R}}^n$ provided with the isotopological structure.

6.3. An isoatlas

Let M be a Haussdorff space. We consider a collection of isocharts $(V_\alpha, \varphi_\alpha)$, $\alpha \in A$, on M with the following properties:

P1 : $\bigcup_{\alpha \in A} V_\alpha = M$

P2 : We consider two isocharts $(V_\alpha, \varphi_\alpha)$ and (V_β, φ_β) with the property $V_\alpha \cap V_\beta \neq \emptyset$. Then we have the isomapping

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$$

which is differentiable.

P 3 : The collection $(V_\alpha, \varphi_\alpha)$, $\alpha \in A$ contains the maximal munber of isocharts satisfying properties P1 and P2.

This collection of isocharts $(V_\alpha, \varphi_\alpha)$, $\alpha \in A$ is called differentiable isoatlas on M.

6.4 Definition of isomanifold

Let $(M, \hat{g})$ be a Haussdorff space on which we consider an isoatlas $\{(V_\alpha, \varphi_\alpha)\}$, $\alpha \in A$. Then the pair $\{M, \{(V_\alpha, \varphi_\alpha), \alpha \in A\}\}$ is called a differentiable isomanifold.

6.5. Different isostructures

Let M be a Haussdorff space. Now, there is the following question: how many different isostructures are there on M. This is related to the basic problem to find different differentiable structures on the topological space M.

If $M = \mathbb{R}^n$, $n \neq 4$, there is only one ([3])

If $M = \mathbb{R}^4$, there are infinite number ([3])

If $M = S^k$, $k \leq 6$, then there is only one

If $M = S^k$, $k \geq 7$, there is a finite number ([7]) .

6.6. Isofunctions on an isomanifold

The isofunctions on an isomanifold M are defined by

$$\hat{C}(M) = \{\hat{f} \;/\; \hat{f} : M \to \hat{\mathbb{R}} \}.$$

In the usual way we can define the differentiability of class C^k of an isofunction $\hat{f}$.

The set of all differentiable isofunctions, that is $k = \infty$, is denoted by $\hat{D}(M)$ which becomes an isoalgebra over the isofield of isoreal numbers.

6.7. Isovector fields on isomanifolds

Let $\hat{D}^0(M)$ be the isoalgebra of the isomanifold M. Every derivation $\hat{X}$ on $\hat{D}^0(M)$ is called isovector on M. Then we have

$$\begin{aligned}
&\hat{X} : \hat{D}^\circ(M) \to \hat{D}^\circ(M)\,, \quad \hat{X} : \hat{f} \to \hat{X}(\hat{f}) \\
&\hat{X} : \hat{a}_1 \hat{f}_1 + \hat{a}_2 \hat{f}_2 \to \hat{X}(\hat{a}_1 \hat{f}_1 + \hat{a}_2 \hat{f}_2) = \hat{a}_1 \hat{X}(\hat{f}_1) + \hat{a}_2 \hat{X}(\hat{f}_2) \\
&\hat{X} : \hat{f}_1 \circ \hat{f}_2 \to \hat{X}(\hat{f}_1 \circ \hat{f}_2) = \hat{f}_1 \hat{X}(\hat{f}_2) + \hat{f}_2 \hat{X}(\hat{f}_1)
\end{aligned} \tag{6.1}$$

where $\hat{a}_1, \hat{a}_2 \in \hat{\mathbb{R}}$, $\hat{f}, \hat{f}_1, \hat{f}_2 \in \hat{D}^1(M)$.

The set of all isovector fields is denoted by $\hat{D}^1(M)$, which is an isomodule over $\hat{D}^0(M)$, that is

$$\hat{D}^1(M) = \{\hat{X} \,/\, \hat{X} : \hat{D}^0(M) \to \hat{D}^0(M), \text{X derivation}\}$$

PPROPOSITION 6.8. *The set of the isovector fields $\hat{D}^1(M)$ can become a Lie algebra and Lie-Santilli algebra.*

P r o o f : From (6.1) we obtain that $\hat{D}^1(M)$ is an isomodule over the $\hat{D}^0(M)$, which is the known isoalgebra.

On $\hat{D}^1(M)$ we can define the Lie bracket [ ] as follows:

$$[\,] : \hat{D}^1(M) \times \hat{D}^1(M) \to \hat{D}^1(M)$$

$$[\,] : (\hat{X}, \hat{Y}) \to [\hat{X}, \hat{Y}] = \hat{X}\,\hat{Y} - \hat{Y}\,\hat{X}$$

where $[\hat{X}, \hat{Y}]\,(\hat{f}) = \hat{X}\,(\hat{Y}(\hat{f})) - \hat{Y}(\hat{X}(\hat{f})), \quad \forall\, \hat{f} \in \hat{D}^0(M)$

It can be easily seen that the Lie bracket [ ] satisfies the anticommutative law and Jacobi's identity and therefore $\hat{D}^1(M)$ with this Lie bracket becomes a Lie algebra.

Now, on $\hat{D}^1(M)$ we define the Santilli bracket

$$[X, Y]_S = \hat{X}\,\hat{T}\,\hat{Y} - \hat{Y}\,\hat{T}\,\hat{X} \tag{6.2}$$

where $\hat{T}$ is a fixed isovector field on M and $\hat{X}, \hat{Y} \in \hat{D}^1(M)$.

It can be easily seen that (6.2) satisfies the anticommutative law and Jacobi's identity. Therefore, $\hat{D}^1(M)$ with Santilli bracket defined by (6.2), is a Lie-Santilli algebra.

6.8. Isoexterior 1-forms

Let $\hat{D}^1(M)$ be the isomodule of isovector fields over $\hat{D}^0(M)$, where M is a differentiable isomanifold.

Let $\hat{D}_1(M) = \hat{D}^1(M)^*$ be the isodual of $\hat{D}^1(M)$, which is again an isomodule over $\hat{D}^0(M)$. That means

$$\hat{D}_1(M) = \{\hat{\omega} \,/\, \hat{\omega} : \hat{D}(M) \to \hat{D}^0(M),\ \omega \text{ a linear form}\}$$

Each element $\hat{\omega} \in \hat{D}_1(M)$ is called isoexterior 1-form.

6.9. Isotensor fields

We consider a differentiablr isomanifold M. From this we obtain the isomodules $\hat{D}^1(M)$ and $\hat{D}_1(M)$ over $\hat{D}^0(M)$. Now, we obtain the Cartesian product

$$\underbrace{\hat{D}^1(M) \times \hat{D}^1(M) \times \dots \times \hat{D}^1(M)}_{\text{s-times}} \times \underbrace{\hat{D}_1(M) \times \hat{D}_1(M) \times \dots \times \hat{D}_1(M)}_{\text{r-times}}$$

Now, we construct the following set:

$\hat{D}^r_s(M) =$

$$= \{\varphi : \underbrace{\hat{D}^1(M) \times \dots \times \hat{D}^1(M)}_{\text{s-times}} \times \underbrace{\hat{D}_1(M) \times \dots \times \hat{D}_1(M)}_{\text{r-times}}, \to \hat{D}^0,\ \varphi \text{ is (r+s)-multilinear mapping}\}$$

Each element $\varphi \in \hat{D}^r_s(M)$ is called (r, s) isotensor field on M. It can be easily seen that $\hat{D}^r_s(M)$ is an isomodule over $\hat{D}^0(M)$.

7. Isoaffine connections

Let M be a differential isomanifold of dimension n. Let $\hat{D}^1(M)$ be the isomodule of the isovector fileds over $\hat{D}^0(M)$.

DEFINITION 7.1. An isoaffine connection on M is a rule $\hat{\nabla}$ which for every isovector field $\hat{X}$ we obtain an isolinear transformation denoted by $\hat{\nabla}_{\hat{X}}$ on $\hat{D}^1(M)$ that is

$$\hat{\nabla}_{\hat{X}} : \hat{D}^1(M) \to \hat{D}^1(M), \quad \hat{\nabla}_{\hat{X}} : \hat{Y} \to \hat{\nabla}_{\hat{X}}\hat{Y}$$

having the following properties:

P1 : $\hat{\nabla}_{\hat{f}\hat{X}+\hat{g}\hat{Y}} = \hat{f}\,\hat{\nabla}_{\hat{X}} + \hat{g}\,\hat{\nabla}_{\hat{Y}}$:

P2 : $\hat{\nabla}_{\hat{X}}(\hat{f}\,\hat{Y}) = \hat{f}\,\hat{\nabla}_{\hat{X}}\hat{Y} + (\hat{X}(\hat{f})\hat{Y}$

where $\hat{f}, \hat{g} \in \hat{D}^0(M)$, $\hat{X}, \hat{Y} \in \hat{D}^1(M)$.

REMARK 7.2. There is the whole theory of isoaffine connections in ([28]).

DEFINITION 7.3. Let M be a differentiable isomanifold on which we consider an isoaffine connection $\hat{\nabla}$. The pair $(M,\hat{\nabla})$ is called isoaffine isomanifold.

REMARK 7.3. The theory in detail for isoaffine connection and issoaffine isomanifolds are given in ([28]).

8. IsoRemannian metric

Let $\hat{g}$ be a symmetric covariant isotensor field of order two on the differentiable isomanifold M. We assume that $\hat{g}$ has the following properties:

$$\mathbf{P1} \quad : \quad \hat{g}(\hat{X}, \hat{Y}) = \hat{g}(\hat{Y}, \hat{X})$$

$$\mathbf{P2} \quad : \quad \hat{g}(\hat{X}, \hat{X}) \geq \hat{0} \tag{8.1}$$

$\forall \hat{X}, \hat{Y} \in \hat{D}^1(M)$ and $\hat{0}$ isozero of $\hat{\mathbb{R}}$.

The relation (8.1) expresses that $\hat{g}$ is positive definite.

$\hat{g}$ can define a metric on M. $\hat{g}$ is called isoRiemannian metric on M.

DEFINITION 8.1. Let $\hat{g}$ be an isoRiemannian metric on the isomanifold $\hat{M}$. The pair $(M, \hat{g})$ is called isoRiemannian isomanifold.

REMARK 8.2. The theory in details for isoRiemannian metrics and isoRiemannian maniflods are given in ([29]).

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STUDIES ON THE CLASSIFICATION OF LIE-SANTILLI ALGEBRAS

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Abstract

Lie-Santilli algebras are a nonlinear, nonlocal-integral and noncanonical generalization of the conventional linear, local-differential and canonical formulation of Lie algebras. In this paper we provide techniques for their classification.

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1. *INTRODUCTION.* Let g be a vector space over the field F. Let V be a set having a law of composition □ with unit e. Let I be an element of V having inverse element T, that means I□T=T□I=O, which is denoted I^{-1}. I is called isounit and T isotopic element [12].

We assume that we can construct the set $\hat{g}=\{\hat{X}:XI/X\in g, I \text{ isounit}\}$.

It can be easily seen that $\hat{g}$ is a vector space over F. On $\hat{g}$ we can define the Santilli brackets, denoted by $[\ ]_s$ and having the property $[\hat{X},\hat{Y}]_s=\hat{X}T\hat{Y}-\hat{Y}T\hat{X}=(XY-YX)I$. If $[\hat{X},\hat{Y}]_s\in\hat{g}$, then $\hat{g}$ is called Lie-Santilli algebra or Lie isotopic algebras [12-17] (see also [4], [19].

The aim of the present paper is to study some properties of the Lie-Santilli algebras and give some criteria for classification of these algebras.

The paper contains five paragraphs. Which can be as follows.

The second paragrph contains an outline of the theory of Lie algebras.

The basic theorems of Lie-Santilli algebras are presented in the third paragraph.

The fourth paragraph presents another approach to the theory of Lie-Santilli algebras.

Some examples of vector spaces, which are simultaneously Lie algebras and Lie-Santilli algebras, are given in the last paragraph.

2. *LIE ALGEBRAS*

Let g be a vector space over a field F of characteristic zero. We assume that on g there exists a second law of composition, denoted by [ ], that means

$$[\ ]:g\text{x}g\to g,$$

$$[\ ]:(X,Y)\to[X,Y]\in g,$$

whish satisfies the following conditions

$$\text{(I)} \qquad [X,Y]=-[X,Y] \quad \text{anticommutative,}$$

$$\text{(II)} \qquad [X, [Y, Z]]+[Z, [X, Y]]+[Y, [Z, [12]=O$$

(Jacobi's is identity).

In this case the law of composition [ ] is called Lie brackets. The vector space g with the Lie brackets is called Lie algebra over a field F.

If a Lie algebra g over the field F has the property

$$[X, Y]=O, \ \forall X, Y\in g,$$

then g is called Abelian.

If g is a Lie algebra over the field F, then from this we can form the following sequences of ideals:

$$D^o g=g, \quad D^1 g, \dots \ D^k g=[D^{k-1} g \quad D^{k-1} g], \dots$$

which is called derived series of g and

$$C^o g=g, \quad C^1 g=[g, C^o g=[g, C^1 g], \dots, C^k g=[g, C^{k-1} g], \dots$$

which is called descending central series of g.

It is obvious that the expresions

$$C^o g=g, \dots, D^k g=[D^{k-1} g \quad D^{k-1} g],$$

$$C^o g = g, \dots, C^k g = [g, C^{k-1} g],$$

are two decreasing sequences of ideals of g, that is,

$$D^o g \supseteq D^1 g \supseteq D^2 g \supseteq \dots \supseteq D^k g \supseteq,$$

$$C^1 g \supseteq C^1 g \supseteq C^2 g \supseteq \dots \supseteq C^k g \supseteq.$$

It can be easily seen that,

$$D^k g \subset C^k g.$$

The Lie algebra g is called nilpotent if $C^p g=O$ for some $p\geq 2$. The Lie algebra g is called solvable if $D^p g=O$ for some $p\geq 2$.

Let g be a Lie algebra over a field F.

The maximal nilpotent ideal n of g is called nilpotent radical. The maximal solvable ideal r of g is called radical. A Lie algebra g is called semi-simple, if its radical is the $\{O\}$. A Lie algebra g is called simple, if it does not have any ideal except $\{O\}$ and the same Lie algebra g.

One of the basic problems of the theory of Lie algebras is their classification.

A fundamental role in this direction is played Levi's theorem, which can be stated as follows.

THEOREM 2.1 : Let g be a Lie algebra over a field F of charactiristic zero. Then g can be written $g=h\oplus r$ as Lie decomposition, where h is semi-simple and r solvable, which is the radical of g.

The classification of semi-simple Lie algebras is contained in the theorem below.

THEOREM 2.2 : Let g be a semi-simple Lie algebra over a field F of charactiristic zero. Then g can be decomposed as follows

$$g=\oplus g_k,$$

where g_k are simple Lie algebras.

The classification of simple Lie algebras is known, as given below.

A_n, B_n, C_n, D_n classical simple Lie algebras and

G_2, F_4, E_6, E_7, E_8 exceptional simple Lie algebras

From Levi's theorem we conclude that in order to have complete classification of all Lie algebras over a field of charactiristic zero we need to classify the solvable Lie algebras.

The classification of solvable Lie algebras is stated in the follow Malzev's theorem.

THEOREM 2.3 : The classification of solvable Lie algebras is reduced to the classification of nilpotent Lie algebras.

From this theorem we conclude that, in order to have the classification of Lie algebras over a field of charactistic zero, we need to solve the problem of classification of nilpotent Lie algebras, which is still a open and very difficult problem. Until now, we have the following result for a nilpotent Lie algebrag of dimension n over a field F of charactiristic O :

If dimg=n≤6, there is complete classification ([1], [2])

If dimg=n=7, there is complete classification ([7])

If dimg=n=8, there is partial classification ([21], [22])

If dimg=n=9, there is partial classification ([20], [23])

For the above classification we have used different techniques, such as the maximal abelian ideals, Jordan representation and others.

The Kac-Moody Lie algebras can also be used for the classification of nilpotent Lie algebras. Using this theory it has been posible to determine all nilpotent Lie algebras of maximal rank and type A, where A is the Cartan matrix one of these simple Lie algebras

$$A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8 .$$

3. *LIE-SANITILLI ALGEBRAS*

Let g be a vector space over a field F of characteric zero. On this vector space we have the composition law + and external law • of g into F. Theorefore, g can be written g(+, •), that is,

$$+: g \times g \to g, \quad +: (X, Y) \to X+Y$$

$$\bullet : F \times g \to g, \quad \bullet : (\lambda, X) \to \lambda \bullet X = \lambda X.$$

Let V be a set with law of composition □, which has an identity e with respect to □. Therofore, we have

$$\square: V \times V \to V, \qquad \square: (I_1, I_2) \to I_1 \square I_2,$$

$$e \square I_1 = I_1 \square e = I_1 \quad \forall\, I_1\, I_2 \in V.$$

We assume that we can define the set

$$\hat{g} = \{\hat{X} = XI \,/ X \in g, \quad I \in V \quad \text{whose inverse is} \quad T = I^{-1}\}$$

I is called isounit and T isotopic element. The set $\hat{g}$ is called isotopic set associated to g by means of the isounit I.

This action, which gives $\hat{g}$ by virtue of g and I, is called isotopy [12].

THEOREM 3.1 : Let $\hat{g}$ by the isotopic set of the vector space g by means of the isounit I. The set $\hat{g}$ can be become a vector space over F.

<u>Proof</u>: We difine a law of composition, denoted by $\hat{+}$ on $\hat{g}$, as follows

$$\hat{+} : \hat{g} \times \hat{g} \to \hat{g},$$

$$\hat{+}:(\hat{X},\hat{Y}) \to \hat{X}\,\hat{+}\,\hat{Y} \ \underline{\text{definition}}\ (X+Y)I .$$

The external law of F into $\hat{g}$, denoted by $\hat{o}$, is defined by

$$\hat{o}: F\times\hat{g} \to \hat{g},$$

$$\hat{o}:(\lambda,\hat{X}) \to \lambda\,\hat{o}\,\hat{X} \ \underline{\text{definition}}\ Io\lambda X=I(\lambda X) .$$

It can be easily proved that $\hat{g}$ with these two laws $(\hat{+}, \hat{o})$ becomes a vector space over F, which is denoded by $\hat{g}\ (\hat{+}, \hat{o})$.
q. e. d.

Proposition 3.2 : Let $\{e_1, \dots , e_n\}$ be a bas is of the vector space g over a field F of characteristic zero. Then $\{\hat{e}_1 = e_1I, \dots , \hat{e}_n = e_nI\}$ is a base of $\hat{g}$.

Proof: If $X\in g$, we can write

$$X=\lambda_1e_1 +\dots +\lambda_ne_n , \tag{3.1}$$

From (3. 1) we obtain

$$\hat{X}=XI=\lambda_1e_1I +\dots+\lambda_ne_nI=\lambda_1\hat{e}_1 +\dots+\lambda_n\hat{e}_n . \tag{3.2}$$

Relation (3.2) implies that the vectors $\{\hat{e}_1=e_1I ,\dots, \hat{e}_n=e_nI\}$ forn a base is of the vector space $\hat{g}$. q. e. d.

Remark 3.3 : It is possible the set V, with the law of composition □ and unit e, to have one of the following forms:

1st : $V\cap g=\{\varnothing\}$;

2nd : $g \subset V$ proper subset of V. In this case the law of composition □ in V and the law of composition in g are same or different and eirther $e=0$ or $e \neq 0$, where 0 is the zero vector of g,in which case we obtain the isounit $I \in V-g$.

3rd : $g=V$. In this case we obtain the isounit $I \neq 0$. It possible g to have another law of composition □ with isounit $e \neq I$. It can be als the case □=+ and $e=I$.

Definition 3.4 : Let $\hat{g}$ be the isotopic vector space of g over a field F of charactiristic zero with isounit $I \in V(□, e)$ and isotopic element T, that is I□T=e. If the law

$$[\hat{X},\hat{Y}]_s=\hat{X}T\hat{Y}-\hat{Y}T\hat{X},$$

is the law of composition in $\hat{g}$, that means

$$[\hat{X},\hat{Y}]_s \in \hat{g},$$

then $\hat{g}$ is called a Lie-Santilli algebra.

This law of composition

$$[\hat{X},\hat{Y}]_s=\hat{X}T\hat{Y}-\hat{Y}T\hat{X} \qquad (3.3),$$

is called Santilli brackets [12].

Proposition 3.5 : **The Lie- Santilli bracket $[\ ]_s$ on the Lie-Santilli algebra $\hat{g}$, satisfiy the following relations**

$$[\hat{X},\hat{Y}]_s=-[\hat{Y},\hat{X}]_s, \qquad (3.4)$$

$$[\hat{X},[\hat{Y},\hat{Z}]_s]_s+[\hat{Z},[\hat{X},\hat{Y}]_s]_s+[\hat{Y},[\hat{Z},\hat{X}]_s]_s=0. \qquad (3.5)$$

<u>Proof:</u> From (3.3) we obtain

$$[\hat{Z},\hat{Y}]_s=\hat{X}T\hat{Y}-\hat{Y}T\hat{X} \qquad (3.6)$$

$$[\hat{Y},\hat{X}]=\hat{Y}T\hat{X}-\hat{X}T\hat{Y}=-(\hat{X}T\hat{Y}-\hat{Y}T\hat{X}) \qquad (3.7)$$

$$\hat{Y}T[\hat{Z},\hat{X}]_s-[\hat{Z},\hat{X}]_sT\hat{Y} \qquad (3.8)$$

which can be written

$$K=\hat{X}T(\hat{Y}T\hat{Z})-\hat{X}T(\hat{Z}T\hat{Y})-\hat{Y}T\hat{Z}\,\hat{X}\,\hat{Z}T\hat{Y}T\hat{X}+$$

$$+\hat{Z}T\hat{X}T\hat{Y}-\hat{Z}T\hat{Y}T\hat{X}-\hat{X}T\hat{Y}T\hat{Z}+\hat{Y}T\hat{X}T\hat{Z}+$$

$$\hat{Y}T\hat{Z}T\hat{X}-\hat{Y}T\hat{X}T\hat{Z}-\hat{Z}T\hat{X}T\hat{Y}+\hat{X}T\hat{Z}T\hat{Y}.$$

Therefore, condition (3.7) for Lie-Santilli algebra is satisfied, which is called the isotopic Jacobi's identity.

Remark 3.6 : Since the two conditions (3.4) and (3.5) are satisfied for a Lie-Santilli algebra, we conclude that the theory of Lie-Santilli algebras are a generalization of Lie algebras.

Now, there is the following a basic question. Under which conditions the isotopic space $\hat{g}$ can become a Lie-Santilli algebra with Santilli brackets?

This can be answered by the following two theorems.

THEOREM 3.7 : **Let g be a Lie algebra of dimension n over a field of charactiristic zero. Let $\hat{g}$ be the isotopic vector space of g with isounit I. Then $\hat{g}$ is a Lie-Santilli algebra with respect to Santilli brackets.**

Proof: It is necessary to prove that

$$[\hat{X},\hat{Y}]_s \in \hat{g} \quad \forall \hat{X}, \hat{Y} \in \hat{g} \tag{3.9}$$

It is known that

$$[X,Y]=X\,Y-Y\,X \in g \quad \forall X, \; Y \in g \tag{3.10}$$

and $\hat{g}=\{\hat{X}=X\,I \,/\, \hat{X} \in g\}$ (3.11)

Therefore the Santilli bracket takes the form

$$[\hat{X},\hat{Y}]_s=\hat{X}T\hat{Y}-\hat{Y}T\hat{X}=X\,I\,T\,Y\,I-Y\,I\,T\,X=$$

$$=XYI-YXI=[X,Y]I \tag{3.12}$$

From (3.10), (3.11) and (3.12) we couclude that

$$[\hat{X},\hat{Y}]_s=[XY]=\hat{g} \tag{3.14}$$

and hence (3.9) is satisfied q. e. d.

Example 3.8 : We consider the Lie algebra A_2, defined by

$$A_2=\left\{X=\begin{pmatrix} a & b \\ c & -a \end{pmatrix} / \; a, b, c \in IR\right\} \tag{3.15}$$

which is three dimensional. Construct the Lie-Santilli algebra using as isounit the following matrix

$$I=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \tag{3.16}$$

Solution. From (3. 16) we obtain

$$T=I^{-1}=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \tag{3.17}$$

The isotopic vector space $\hat{A}_2$ is defined by

$$\hat{A}_2=\left\{\hat{X}=XI \doteq \begin{pmatrix}a+b & -a+b\\ -a+c & -a-c\end{pmatrix}\right\} \tag{3.18}$$

which is a Lie-Santilli algebra with the Santilli bracket

$$[\hat{X},\hat{X}_1]_s=\hat{X}T\hat{X}_1-\hat{X}_1T\hat{X}=(X X_1-X_1 X)I=[X X_1]I=$$

$$=.\left[\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ c_1 & -a_1\end{pmatrix}-\begin{pmatrix}a_1 & b_1\\ c_1 & -a_1\end{pmatrix}\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\right]\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}=$$

$$\left[\begin{pmatrix}aa_1+bc_1 & ab_1-ba_1\\ ca_1-ac_1 & cb_1+aa_1\end{pmatrix}-\begin{pmatrix}a_1a+b_1c & a_1b-b_1a\\ c_1a-a_1c & c_1b+aa_1\end{pmatrix}\right]\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}bc_1-c_1b & 2(ab_1-ba_1)\\ 2(ca_1-ac_1)- & (bc_1-c_1b)\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}cb_1-c_1b+2(ab_1-ba_1) & -(bc_1-c_1b)+2(ab_1-ba_1)\\ 2(ca_1-ac_1)-(bc_1-c_1b) & -2(ca_1-ac_1)-(bc_1-c_1b)\end{pmatrix}.$$

Remark 3.7 : In this paper we have used the lifting from a Lie to a Lie-Santilli algebra

$$X\rightarrow\hat{X}=XI,$$

called "Klimyk rule") [16]. Its main characteristics are those of preserving structure constants and weights of the original Lie algebra, as shown by the rule for an order set of generators $X=\{X_i\}$, I=1, 2, ... , n,

$$X_iX_j-X_jX_i=(\hat{X}_iT\hat{X}_j-\hat{X}_jT\hat{X}_i)T=C^k{}_{ij}X_k=(C^k{}_{ij}\hat{X}_k)T.$$

The elimination of the factor T then yields the isotopic rules $\hat{X}_iT\hat{X}_j-\hat{X}_jT\hat{X}_i=C^k{}_{ij}\hat{X}_k$ uniquely and unambiguously from the original

rules $X_iX_j-X_jX_i=C^k{}_{ij}X_k$. The preservation of the original weights on a basis, can be seen, for insatnce, from the fact that the numerical valures of a diagonal element, say, X_k, are preserved identically under the above rule.

It is important to point out that the above lifting is the simplest possible one from a Lie to a Lie-Santilli algebra. In fact, the most general lifting is characterized by transforms which do not preserve the original weights. As an illustration, suppose that the original Lie algebra g is characterized by an ordered set of operators $X=\{X_i\}$, =1, 2, ... , n, defined on a conventional Hilbert space. Then, the most general possible map from g to the Lie-Santilli algbra $\hat{g}$ is characterized by the nonunitary transform

$$U\ U^+=I\neq 1,\quad T=(U\ U^+)^{-1}$$

under which we have

$$U[X_iX_j-X_jX_i)U^+=\hat{X}_iT\hat{X}_j-\hat{X}_j\hat{X}_i=U(C^k{}_{ij}X_k)U^+=C^k{}_{ij}\hat{X}_k,$$

$$\hat{X}_i=UX_iU^+.$$

As one can see, the structure are again preserves, thus preserving the local isomorphisms $g=\hat{g}$ for positive definite isount I. However, while the original weight of a Lie algebra g are preserved under the rele $X\rightarrow\hat{X}=XI$, the same weights are not evidently preserved under the more general lifting $X\rightarrow\hat{X}=UXU^+\neq 1$[16].

In the latter case, the Lie-Santilli algebras becomes nontrivial generalization of Lie algebras because, the representation theory of Lie algebras is linear, local and Hamiltonian, while that of the Lie-Santilli algebras is in general nonlinear, nonlocal and nonhamiltonian [16].

We finally remark that the definition of a Lie-Santilli algebra over a conventional field F as done in this paper is correct only under the Klimyk rele. In fact, for the general case, a Lie-Santilli, algebra must be defined for consistency over isofields [15] which are the image of the ordinary fields when reconstructed with respect to the isounit I (see [4] and [16] for details).

4. NEW APROACH OF LIE-SANTILLI ALGEBRAS

There is the following problem. We assumet that g is a vector space over the field F, which is not a Lie algebra with the Lie brackets.

$$[XY] \notin g, \quad \forall X, Y \in g \qquad (4.1)$$

Is it possible to find Santilli bracket $[\ ]_s$ on g such that g with these brackets $[\ ]_s$ becomes a Lie-Santilli algebra?

We assume that there is a set $V \supseteq g$ with law of composition $\square$ having the identity element e. Let I be an element of V having invense element T, that is,

$$I \square T = T \square I = e. \qquad (4.2)$$

We suppose that we can construct the set

$$\hat{g} = \{ \hat{X} = XI / X \in g \}. \qquad (4.3)$$

It has been proved that $\hat{g}$ is a vector space over F.

We assume that we can define the following bracket laws

$$[X, Y] = XY - YX, \quad [X, Y]_s = XI\,Y - YI\,X, \; [X, Y]_s = XTY - YTX \qquad (4.4)$$

We know that $[X, Y]$ is not law of composition on g. It is however possible the $[X, Y]_s$ and $[X, Y]_s$ are laws of composition on g.

We also assume the laws

$$[\hat{X},\hat{Y}]=\hat{X}\hat{Y}-\hat{Y}\hat{X},\ [\hat{X},\hat{Y}]_s=\hat{X}I\hat{Y}-\hat{Y}I\hat{X},\ [\hat{X},\hat{Y}]_s=\hat{X}T\hat{Y}-\hat{Y}T\hat{X}. \quad (4.5)$$

It is possible that some of them define a law of composition on $\hat{g}$.

THEOREM 4.1 : The laws, defined by (4.4) and (4.5) satisfy the condition of anticommutativity and the Jacobi's identity.

Proof: If we assume the brackets

$$[\hat{X},\hat{Y}]_s=\hat{X}I\hat{Y}-\hat{Y}I.\hat{X},$$

then using the same techique as in proposition (3.5), the following relations can be proved.

$$[\hat{X},\hat{Y}]_5=[\hat{Y},\ \hat{X}]_5,$$

$$[\hat{X},[\hat{Y},\hat{Z}]_s]_s+[\hat{Z},[\hat{X},\hat{Y}]_s]_s+[\hat{Y},[\hat{Z},\ \hat{X}]_s]_s=0.$$

The same results are valid for the other bracket laws q. e. d.

Example 4.2 : We consider the vector space

$$g=\{X=\begin{pmatrix} a & b \\ b & d \end{pmatrix}\ /\ a,b,d,\in IR\}$$

We obtain as isounit the matrix

$$I=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can prove that g is a Lie-Santilli algebra with the following Santilli brackets

$$[X\ Y]_s = XIY - YIX$$

$$\hat{g} = \{\ \hat{X}I = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -b \end{pmatrix} \} = A_2$$

which characterize the classical Lie algebra A_2.

Therefore by means of theorem 4.1 is a Lie-Santilli algebra with Santilli brackets

$$[XY]_s = XIY - YIX.$$

5. LIE AND LIE-SANTILLI ALGEBRAS

In this section we show that a Lie algebra g , can also form at the same time a Lie-Santilli algebra with a special isounit. some.

Example 5.1 : Let gl (n, IR) be the set of all n square matrices. Conctruct different bracket laws and examine which of them are laws of composition.

Solution. It is known that gl (n, IR) is a vector space over IR and its dimension is n^2. This vector space with Lie bracket

$$[A, B] = AB - BA$$

becomes a Lie algebra.

We consider a non-singular matrix T, that is $D(T) \neq 0$, and hence there exists the inverse $S = T^{-1}$ of T. We construct the set

$$gl(\hat{n}, IR)_T = \{\ \hat{A} = AT / A \in gl(n, IR)\}$$

which is a subset of gl(n, IR), that is

$$gl(\hat{n}, IR)_T \subseteq gl(n, IR)$$

$gl(\hat{n}, IR)$ is a Lie-Santilli algebra with Santilli brackets,

$$[\hat{A}, \hat{B}]_S=\hat{A}T\hat{B}-\hat{B}T\hat{A}=[AB]I\in gl(n, IR)$$

This example has infinite number of solution.

Example 5.2 : We consider the Lie algebra

$$B_1=0(3)=\left\{A=\begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix} / \ a, \ b, \ c\in IR\right\},$$

and obtain as isounit I the matrix

$$I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},$$

whose invere T is the same matrix, that is $T=I^{-1}=I$. Prove that $\hat{B}_1$ is semeltaneuously Lie algebra and a Lie-Santilli algebra .
Solution. The isoset of B_1 is defined by

$$\hat{B}_1=\hat{O}(3)=\left\{\hat{A}=AI=\begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}=\begin{pmatrix}0 & a & -b\\ -a & 0 & -c\\ -b & -c & 0\end{pmatrix}\right\}$$

It can be easily seen that this set is a Lie algebra, that means

$$[\hat{A}-\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}\in\hat{B}_1,$$

and at the same time Lie-Santilli algebra

$$[\hat{A}-\hat{B}]_S=\hat{A}T\hat{B}-\hat{B}T\hat{A}\in\hat{B}_1.$$

This Lie algebra is known and denoted by

$$0(2, 1) = \hat{0}(3)$$

with isounit I the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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